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Algorithms & Data Structures — Exercise sheet 1 — HS 24

The solutions for this sheet are submitted on Moodle until 29 September 2024, 23:59.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

You can use results from previous parts without solving those parts.

Exercise 1.1 *Mathematical induction* **(2 points)**.

(a) Prove by mathematical induction that for every integer $n \geq 0$,

$$1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

(b) Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 6a_n - 2$ for $n \geq 1$. Determine the smallest positive integer m such that $a_m > 2^{2m}$. Then, prove by induction that $a_n \geq 2^{2n}$ for all integers $n \geq m$. (If you are unable to determine m, use $m = 10$. You may assume that $a_{10} \geq 2^{20}$). In your solution, you should address the base case, the induction hypothesis and the induction step.

Exercise 1.2 *Sums of powers of integers.*

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \leq n^4$.

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \geq \frac{1}{2^4} \cdot n^4$.

Hint: *Consider the second half of the sum, i.e., $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3$. How many terms are there in this sum? How small can they be?*

Together, these two inequalities show that $C_1 \cdot n^4 \leq \sum_{i=1}^{n} i^3 \leq C_2 \cdot n^4$, where $C_1 = \frac{1}{2^4}$ and $C_2 = 1$ are two constants independent of n. Hence, when n is large, $\sum_{i=1}^{n} i^3$ behaves "almost like n^4" up to a constant factor.

(c)* Show that parts (a) and (b) generalise to an arbitrary $k \geq 4$, i.e., show that $\sum_{i=1}^{n} i^k \leq n^{k+1}$ and that $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$ holds for any $n \in \mathbb{N}_0$.

Exercise 1.3 *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m \to \infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements with a computation.

(a) $f(m) = 10m^5 + 90m^4$ grows asymptotically slower than $g(m) = 100m^5$.

(b) $f(m) = 80 \cdot 2^m \log(m) - 2^m$ grows asymptotically slower than $g(m) = 5 \cdot 2^m \log(m)^2$.

(c) $f(m) = \log(m^3)$ grows asymptotically slower than $g(m) = \log(m)^3$.

(d) $f(m) = 4^{(m^2+m+1)}$ grows asymptotically slower than $g(m) = 2^{(3m^2)}$.

(e)* If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any* $a, b : \mathbb{N} \to \mathbb{R}^+$, *if* $\lim_{m\to\infty} a(m) = A$ *and* $\lim_{m\to\infty} b(m) = B$, *then* $\lim_{m\to\infty} a(m)b(m) = AB$.

(f)* If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

Exercise 1.4 *Proving Inequalities.*

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}, \quad n \geq 1.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

(b)* Replace $3n + 1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

However, as argued above in the exercise statement, the inequality is still true. We are just not able to prove it directly via mathematical induction.