Eidgenössische
Technische Hochschule
Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Federal Institute of Technology at Zurich

Departement of Computer Science
Johannes Lengler, David Steurer
Kasper Lindberg, Lucas Slot, Hongjie Chen, Manuel Wiedmer

23 September 2024

Algorithms & Data Structures — Exercise sheet 1 — HS 24

The solutions for this sheet are submitted on Moodle until 29 September 2024, 23:59.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

You can use results from previous parts without solving those parts.

The solutions are intended to help you understand how to solve the exercises and are thus more detailed than what would be expected at the exam. All parts that contain explanation that you would not need to include in an exam are in grey.

Exercise 1.1 *Mathematical induction* **(2 points)**.

(a) Prove by mathematical induction that for every integer $n \geq 0$,

$$1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

Solution:

Base Case.
Let $n = 0$. Then,

$$1 = (0 + 1)^2,$$

so the property holds for $n = 0$.

Induction Hypothesis.
Assume that the property holds for some integer $k \geq 0$, that is,

$$1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2.$$

Induction Step.
We must show that the property also holds for $k + 1$. Let us add $2(k + 1) + 1$ to both sides of the induction hypothesis. We get

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2k + 1) + (2(k + 1) + 1) &= (k + 1)^2 + (2(k + 1) + 1) \\ &= (k^2 + 2k + 1) + (2k + 3) \\ &= k^2 + 4k + 4 \\ &= (k + 2)^2 = ((k + 1) + 1)^2. \end{aligned}$$

By the principle of mathematical induction, $1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2$ is true for any integer $n \geq 0$.

Guidelines for correction:

Award 1 point for a fully correct solution. Award $1/2$ point if the solution is mostly correct but contains a minor mistake in the computation. Award no points if there is a mistake in the logical structure of the induction proof.

(b) Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 6a_n - 2$ for $n \geq 1$. Determine the smallest positive integer m such that $a_m > 2^{2m}$. Then, prove by induction that $a_n \geq 2^{2n}$ for all integers $n \geq m$. (If you are unable to determine m, use $m = 10$. You may assume that $a_{10} \geq 2^{20}$). In your solution, you should address the base case, the induction hypothesis and the induction step.

Solution:

Writing out the first few terms of the recursive formula, we get $a_1 = 2, a_2 = 10, a_3 = 58, a_4 = 346$. On the other hand, $2^{2\cdot 1} = 4, 2^{2\cdot 2} = 16, 2^{2\cdot 3} = 64, 2^{2\cdot 4} = 256$. So $m = 4$ is the smallest positive integer where $a_m \geq 2^{2m}$.

We prove that $a_n \geq 2^{2n}$ for all integers $n \geq m = 4$ using induction.

Base Case.
Our base case is $n = m = 4$. As we have seen above,

$$a_4 = 346 \geq 256 = 2^{2\cdot 4}.$$

So the inequality holds for $n = 4$.

Induction Hypothesis.
We now assume that it is true for $n = k$, i.e., $a_k \geq 2^{2k}$.

Induction Step.
We want to prove that it is also true for $n = k + 1$. Using the recursive formula, and the induction hypothesis, we get

$$a_{k+1} \overset{\text{RF}}{=} 6a_k - 2 \overset{\text{IH}}{\geq} 6 \cdot 2^{2k} - 2 = 4 \cdot 2^{2k} + 2 \cdot 2^{2k} - 2 \geq 4 \cdot 2^{2k} = 2^2 \cdot 2^{2k} = 2^{2(k+1)}.$$

Hence, it is true for $n = k + 1$.

By the principle of mathematical induction, we conclude that $a_n \geq 2^{2n}$ is true for any integer $n \geq m = 4$.

Guidelines for correction:

Award 1 point for a fully correct solution. Award $1/2$ point if the solution is mostly correct but contains a minor mistake in the computation (this includes a mistake in determining m or not determining m at all and using $m = 10$). Award no points if there is a mistake in the logical structure of the induction proof.

Exercise 1.2 *Sums of powers of integers.*

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \leq n^4$.

Solution:

As all terms in the sum are at most n^3, we have

$$\sum_{i=1}^{n} i^3 \leq \sum_{i=1}^{n} n^3 = n \cdot n^3 = n^4.$$

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \geq \frac{1}{2^4} \cdot n^4$.

Hint: *Consider the second half of the sum, i.e.,* $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3$. *How many terms are there in this sum? How small can they be?*

Solution:

We have

$$\sum_{i=1}^{n} i^3 \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3 \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^3 = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left(\frac{n}{2}\right)^3.$$

By definition of $\lceil \cdot \rceil$, we have $\left\lceil \frac{n}{2} \right\rceil - 1 \leq \frac{n}{2}$, hence, $n - \left\lceil \frac{n}{2} \right\rceil + 1 \geq \frac{n}{2}$. Hence,

$$\sum_{i=1}^{n} i^3 \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^3 = \frac{1}{2^4} \cdot n^4.$$

The idea of the above is as follows. Similar to part (a), we want to uniformly bound the terms of the sum. However, if we do this for all terms, the best bound that is true (for all terms in the sum) is 1, which does not help proving the statement (we would only get a lower bound of n). Thus, we restrict to half the terms. This allows us (roughly, up to using $\left\lceil \frac{n}{2} \right\rceil$) to say that we have $\frac{n}{2}$ term that are at least $\left(\frac{n}{2}\right)^3$. Formalising this idea yields the above.

Together, these two inequalities show that $C_1 \cdot n^4 \leq \sum_{i=1}^{n} i^3 \leq C_2 \cdot n^4$, where $C_1 = \frac{1}{2^4}$ and $C_2 = 1$ are two constants independent of n. Hence, when n is large, $\sum_{i=1}^{n} i^3$ behaves "almost like n^4" up to a constant factor.

(c)* Show that parts (a) and (b) generalise to an arbitrary $k \geq 4$, i.e., show that $\sum_{i=1}^{n} i^k \leq n^{k+1}$ and that $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$ holds for any $n \in \mathbb{N}_0$.

Solution:

Similar to part (a), to show that $\sum_{i=1}^{n} i^k \leq n^{k+1}$ we note that all terms in the sum are at most n^k. Thus, we get that

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}.$$

To show that $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$, we consider again the second half of the sum, i.e. only $i \geq \left\lceil \frac{n}{2} \right\rceil$. We get that

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left(\frac{n}{2}\right)^k.$$

As before, we have that $\left\lceil \frac{n}{2} \right\rceil - 1 \leq \frac{n}{2}$ and $n - \left\lceil \frac{n}{2} \right\rceil + 1 \geq \frac{n}{2}$. Hence, we can conclude that

$$\sum_{i=1}^{n} i^k \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} \cdot n^{k+1}.$$

Exercise 1.3 *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements with a computation.

(a) $f(m) = 10m^5 + 90m^4$ grows asymptotically slower than $g(m) = 100m^5$.

Solution:

False, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{10m^5 + 90m^4}{100m^5} \\ &= \lim_{m\to\infty} \frac{10}{100} + \frac{90}{100m} = \frac{1}{10} + 0 > 0.\end{aligned}$$

(b) $f(m) = 80 \cdot 2^m \log(m) - 2^m$ grows asymptotically slower than $g(m) = 5 \cdot 2^m \log(m)^2$.

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{80 \cdot 2^m \log(m) - 2^m}{5 \cdot 2^m \log(m)^2} \\ &= \lim_{m\to\infty} \frac{80}{5\log m} - \frac{1}{5\log(m)^2} = 0 - 0 = 0.\end{aligned}$$

(c) $f(m) = \log\left(m^3\right)$ grows asymptotically slower than $g(m) = \log\left(m\right)^3$.

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{\log(m^3)}{\log(m)^3} \\ &= \lim_{m\to\infty} \frac{3\log(m)}{\log(m)^3} \\ &= \lim_{m\to\infty} \frac{3}{\log(m)^2} = 0.\end{aligned}$$

(d) $f(m) = 4^{(m^2+m+1)}$ grows asymptotically slower than $g(m) = 2^{(3m^2)}$.

Solution:

True, since

$$\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{4^{(m^2+m+1)}}{2^{(3m^2)}} \\
&= \lim_{m\to\infty} \frac{2^{2(m^2+m+1)}}{2^{(3m^2)}} \\
&= \lim_{m\to\infty} \frac{2^{2m^2+2m+2}}{2^{(3m^2)}} \\
&= \lim_{m\to\infty} 2^{2m^2+2m+2-3m^2} \\
&= \lim_{m\to\infty} 2^{-m^2+2m+2} = 0,
\end{aligned}$$

as $\lim_{m\to\infty} -m^2+2m+2 = -\infty$.

(e)* If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any $a, b : \mathbb{N} \to \mathbb{R}^+$, if $\lim_{m\to\infty} a(m) = A$ and $\lim_{m\to\infty} b(m) = B$, then $\lim_{m\to\infty} a(m)b(m) = AB$.*

Solution:

True, since

$$\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{h(m)} &= \lim_{m\to\infty} \frac{f(m)g(m)}{h(m)g(m)} \\
&= \lim_{m\to\infty} \frac{f(m)}{g(m)} \cdot \frac{g(m)}{h(m)} \\
&= \lim_{m\to\infty} \frac{f(m)}{g(m)} \cdot \lim_{m\to\infty} \frac{g(m)}{h(m)} = 0.
\end{aligned}$$

(f)* If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

Solution:

False, consider $f(m) = 1/m^2$, $g(m) = 1/m$ and $h(m) = m^3$. They satisfy the conditions, but f does not grow slower than $g(h(m)) = 1/m^3$.

Guidelines for correction:

Award 1 point if all subtasks (a, b, c, d) are solved correctly. Award $1/2$ point if at least two subtasks are solved correctly.

Exercise 1.4 *Proving Inequalities.*

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \le \frac{1}{\sqrt{3n+1}}, \quad n \ge 1.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

Solution:

Base Case.
For $n = 1$ we have

$$\frac{1}{2} \leq \frac{1}{\sqrt{4}},$$

which is even an equality.

Induction Hypothesis.
Now we assume that it is true for $n = k$, i.e.,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}.$$

Induction Step.
We will prove that it is also true for $n = k+1$, that is we want to show that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}.$$

Plugging in the induction hypothesis, we can upper bound the left hand side of the above inequality as follows,

$$\left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2k-1}{2k}\right) \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2}\,.$$

Then it is sufficient (but might not necessary) to prove

$$\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}\,,$$

which is equivalent to prove

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k+1}}{\sqrt{3k+4}}\,.$$

We can simplify this further as follows:

$$\begin{aligned}
\frac{2k+1}{2k+2} \leq \sqrt{\frac{3k+1}{3k+4}} &\iff \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k+1}{3k+4} \\
&\iff (4k^2+4k+1)(3k+4) \leq (4k^2+8k+4)(3k+1) \\
&\iff 12k^3+28k^2+19k+4 \leq 12k^3+28k^2+20k+4 \\
&\iff 0 \leq k
\end{aligned}$$

This last statement is true since $k \in \mathbb{N}$. Hence, the inequality is also true for $n = k+1$.

Note that for the first equivalence we need that all terms involved are positive, which is true since we have $k \in \mathbb{N}$.

By the principle of mathematical induction, we conclude that the inequality is true for any positive integer n.

(b)* Replace $3n + 1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

Solution:

Sometimes it is easier to prove more than less. This simple approach does not work for the weaker inequality as we are using a weaker (and insufficiently so!) induction hypothesis in each step.

If we try to do the same proof as above, we need to show in the induction step that

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k}}{\sqrt{3k+3}}.$$

Continuing as above, we get that we want to show that

$$\begin{aligned}
\frac{2k+1}{2k+2} \leq \sqrt{\frac{3k}{3k+3}} &\iff \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k}{3k+3} \\
&\iff (4k^2+4k+1)(3k+3) \leq (4k^2+8k+4)(3k) \\
&\iff 12k^3+24k^2+15k+3 \leq 12k^3+24k^2+12k \\
&\iff 3k+3 \leq 0,
\end{aligned}$$

which is not true.

However, as argued above in the exercise statement, the inequality is still true. We are just not able to prove it directly via mathematical induction.