D-INFK
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Linear Algebra

HS 2024

Assignment 1

Submission Deadline: **01 October, 2024** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Lines in $\mathbb{R}^m$ (in-class) (★★☆)

a) Let $\mathbf{0} \in \mathbb{R}^m$ denote the vector whose entries are all zero. We say that a set L is a line in $\mathbb{R}^m$ if and only if there exists $\mathbf{w} \in \mathbb{R}^m$ with $\mathbf{w} \neq \mathbf{0}$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. Let now L be a line in $\mathbb{R}^m$ and let $\mathbf{u}$ be an arbitrary non-zero element of L. Prove that $L = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$.

b) For two lines L_1 and L_2 in $\mathbb{R}^m$, prove that we have either $L_1 \cap L_2 = \{\mathbf{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$.

2. Hyperplanes (hand-in) (★★☆)

We call a set of vectors $H \subseteq \mathbb{R}^m$ a hyperplane of $\mathbb{R}^m$ if and only if there exists a non-zero vector $\mathbf{d} \in \mathbb{R}^m$ such that $H = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$.

a) Consider an arbitrary line L in $\mathbb{R}^2$ (according to the definition in Exercise 1). Prove that L is a hyperplane, i.e. find a vector $\mathbf{d} \neq \mathbf{0}$ such that

$$L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}.$$

b) Consider the following set $S = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = c\}$ for some vector $\mathbf{d} \in \mathbb{R}^m$ and some non-zero constant $c \in \mathbb{R}$. Observe that S is not a hyperplane of $\mathbb{R}^m$ because c is not zero. Finally, we also introduce the set

$$S' = \{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \\ 1 \end{bmatrix} \in \mathbb{R}^{m+1} : \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in S\}$$

which is a subset of $\mathbb{R}^{m+1}$. Prove that S' is a subset of a hyperplane of $\mathbb{R}^{m+1}$.

3. Cauchy-Schwarz inequality (★★☆)

Consider an arbitrary vector $\mathbf{v} \in \mathbb{R}^m$.

a) Prove the inequality

$$\sum_{i=1}^{m} v_i \leq \sqrt{m} \, \| \mathbf{v} \| .$$

b) Prove the inequality

$$\sum_{i=1}^{m} \sqrt{i} v_i \leq m \| \mathbf{v} \| .$$

4. Linear independence (★☆☆)

a) Are the following three vectors in $\mathbb{R}^3$ linearly independent?

$$\mathbf{u} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

b) Are the following three vectors in $\mathbb{R}^4$ linearly independent?

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

5. Linear independence (★★☆)

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m \in \mathbb{R}^m$ be the standard unit vectors in $\mathbb{R}^m$. Consider the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^m$ with $\mathbf{v}_i := \mathbf{e}_i + \mathbf{e}_{i+1}$ for all $i \in \{1, 2, \ldots, m-1\}$ and $\mathbf{v}_m := \mathbf{e}_m + \mathbf{e}_1$.

For example, we get

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

in the case $m = 3$, and

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

in the case $m = 4$.

a) Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly dependent if m is even.

b) Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent if m is odd.

6. Angle between two vectors (★★★)

Consider two non-zero vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$ in $\mathbb{R}^3$ with $x + y + z = 0$. Determine the value of $\cos(\alpha)$ where α denotes the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$. You are not required to compute (or look up) α, but you are of course welcome to do so.

7. **Challenge 1.6** (★★★)

This exercise asks you to solve Challenge 1.6 from the lecture notes (check it out for some guiding questions).

Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be arbitrary vectors in $\mathbb{R}^2$ and assume that $\mathbf{v} \neq \mathbf{0}$ and that $\mathbf{w} \neq \lambda\mathbf{v}$ for all $\lambda \in \mathbb{R}$. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ be arbitrary. Prove that $\mathbf{u}$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.