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Assignment 12

Submission Deadline: **17 December, 2024** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get bonus points and feedback from your TA for Exercise 3 by handing in your solution as pdf via Moodle before the deadline.

1. Eigenvalues and eigenvectors (in-class) (★☆☆)

a) Let $A \in \mathbb{R}^{2\times 2}$ be such that $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ for all $x, y \in \mathbb{R}$. Find all real eigenvalues of A. For each real eigenvalue, find a corresponding real eigenvector of A.

b) Construct a square matrix A with eigenvalues $0, 1, 2$. Furthermore, these should be the only eigenvalues of A.

c) Construct a square matrix B with eigenvalues $0, 1, 2$ such that B is not a diagonal matrix. As before, these should be the only eigenvalues of B.

2. Eigenvalues and eigenvectors of AB and BA (in-class) (★★☆)

Let A, B be two matrices in $\mathbb{R}^{n\times n}$.

a) Let $\lambda \in \mathbb{R}$ be a real eigenvalue of AB. Prove that λ is a real eigenvalue of BA.

b) Assume that B is invertible and that AB has a complete set of real eigenvectors (according to Definition 7.1.20). Prove that BA has a complete set of real eigenvectors.

c) Assume that both A and B are invertible. Prove that AB has a complete set of real eigenvectors if and only if BA has a complete set of real eigenvectors.

d) Can you find an example of two matrices A and B such that BA has a complete set of real eigenvectors, but AB does not have a complete set of real eigenvectors?

3. Recursively defined sequences (bonus, hand-in) (★★☆)

Consider the sequence of numbers given by $a_0 = 1, a_1 = 1$ and $a_n = -a_{n-1} + 6a_{n-2}$ for $n \geq 2$. Find $\alpha, \beta \in \mathbb{R}$ such that $a_n = \frac{4}{5}\alpha^n + \frac{1}{5}\beta^n$ for all $n \in \mathbb{N}_0$. Prove your answer.

***Hint**: Define $\mathbf{v}_i := \begin{bmatrix} a_{i+1} \\ a_i \end{bmatrix}$ and find a matrix $A \in \mathbb{R}^{2\times 2}$ satisfying $A\mathbf{v}_i = \mathbf{v}_{i+1}$ for all $i \in \mathbb{N}_0$. Consider diagonalizing A.*

4. Diagonalization and similarity (★★☆)

This exercise includes Challenge 49 and Challenge 50. Note that the different subtasks are not necessarily building on each other.

Let A, B, C be matrices in $\mathbb{R}^{n\times n}$.

a) Assume that A and B are similar. Prove that the characteristic polynomials $\det(A - zI)$ and $\det(B - zI)$ are equal (where z is the variable of the polynomial).

b) Assume that A has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Assume further that B has the same eigenvectors as A (i.e. every eigenvector of A is an eigenvector of B and vice versa). Prove that $AB = BA$.

c) Assume that A and B are similar. Further assume that B and C are similar. Prove that A and C are similar.

d) Assume that A and B have the same n distinct real eigenvalues. Prove that A and B are similar.

e) Assume that A and B are similar. Prove that they have the same real eigenvalues. Note that you are not allowed to use Proposition 7.2.4 for this exercise.

5. Eigenvalues of rotations and reflections (★★☆)

This exercise is related to Challenge 39 and Example 7.1.16.

Consider two linear transformations $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ and $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ where Φ is a rotation around the y-axis (i.e. the axis given by $\mathbf{e}_2$) by $\frac{\pi}{3}$, and Ψ is the reflection through the plane $P = \{\begin{bmatrix} x & y & z \end{bmatrix}^\top \in \mathbb{R}^3 : 3x + 4y = 0\}$.

Note that the rotation is specified according to the right-hand rule in a right-handed coordinate system[1]: if you imagine your right thumb to be the vector $\mathbf{e}_2$, then slightly curling the other fingers on your right hand will give you the direction of the rotation. For this to uniquely describe the rotation, it is important to specify that we are thinking of a right-handed coordinate system.

a) Find a matrix $A \in \mathbb{R}^{3\times 3}$ such that $A\mathbf{x} = \Phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

b) Find a matrix $B \in \mathbb{R}^{3\times 3}$ such that $B\mathbf{x} = \Psi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

c) Prove that both A and B are orthogonal.

d) Find one real eigenvalue and a corresponding real eigenvector of A.

Hint: You might not have to calculate them. It's valid to guess them and verify that they are indeed an eigenvalue-eigenvector pair.

e) Find two distinct real eigenvalues of B, and a corresponding real eigenvector for each of them.

Hint: You might not have to calculate them. It's valid to guess them and verify that they are indeed eigenvalue-eigenvector pairs.

[1] https://en.wikipedia.org/wiki/Right-hand_rule

6. Orthonormal eigenvectors (★★★)

Consider the three orthonormal vectors

$$\mathbf{v}_1 = \frac{1}{9}\begin{bmatrix} 1 \\ 8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{9}\begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{9}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

with corresponding scalars $\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0.$

a) Let $A \in \mathbb{R}^{3\times 3}$ be a matrix that has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, i.e. $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for all $i \in [3]$. What is the value of $\det(A)$?

b) Prove that the vector $\begin{bmatrix} 16 & 2 & 8 \end{bmatrix}^\top$ belongs to the nullspace of A.

c) Find such a matrix A.

7. Application: population with three age groups (★★☆)

Consider a species with three age groups. Let x_t denote the number of individuals in the first age group at time $t \in \mathbb{N}_0$, y_t the number of individuals in the second age group at time $t \in \mathbb{N}_0$, and z_t the number of individuals in the third age group at time $t \in \mathbb{N}_0$. Moreover, assume the following:

- At every point in time, half of the individuals from the first age group make it to the second age group (the other half dies).
- At every point in time, a third of the individuals from the second age group make it to the third age group (the other two thirds die).
- At every point in time, each individual in the second group leads to one newborn individual (in group one).
- At every point in time, each individual in the third group leads to three newborns (in group one).

For $t \in \mathbb{N}_0$, we use the vector $\mathbf{v}_t = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^\top$ to summarize the current population. Hence, according to the description above, we have $\mathbf{v}_{t+1} = A\mathbf{v}_t$ where

$$A = \begin{bmatrix} 0 & 1 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

Is there a choice for the initial population $\mathbf{v}_0 \in \mathbb{R}^3$ such that the population will be stable over time (i.e. $\mathbf{v}_t = \mathbf{v}_{t+1}$ for all $t \in \mathbb{N}_0$)? If possible, make sure that all entries of $\mathbf{v}_0$ are non-negative since that seems more realistic.