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Assignment 13

Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

There will be no hand-in for this assignment. Solutions will be published on December 22.

Exercises

1. A positive semidefinite matrix (in-class) (★☆☆)

Let $n \in \mathbb{N}^+$ and let $S \in \mathbb{R}^{n\times n}$ such that $S^\top = -S$. Prove that $-S^2$ is symmetric and positive semidefinite.

2. A positive definite matrix (★★☆)

Let $n \in \mathbb{N}^+$ be arbitrary and consider the matrix $A \in \mathbb{R}^{n\times n}$ defined as

$$A := (n-1)I + B$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix and $B \in \mathbb{R}^{n\times n}$ satisfies $B_{ij} = 1$ for all $i, j \in \{1, 2, \dots, n\}$ (i.e. all entries of B are 1). Prove that A is positive definite.

3. Pseudoinverse via SVD (★★☆)

This exercise includes parts of Challenge 56.

Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank r with singular value decomposition (SVD) $A = U_r \Sigma_r V_r^\top$ with $U_r \in \mathbb{R}^{m\times r}$, $\Sigma_r \in \mathbb{R}^{r\times r}$, and $V_r \in \mathbb{R}^{n\times r}$. Recall that A has a pseudoinverse $A^\dagger$. Note that Σ_r is invertible since it is a square diagonal matrix with non-zero entries on its diagonal. Prove that $A^\dagger = V_r \Sigma_r^{-1} U_r^\top$.

4. Least squares via SVD (★★☆)

This exercise includes parts of Challenge 56.

In this task, we derive the solution of the least squares method using the singular value decomposition. Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$ and $\mathbf{b} \in \mathbb{R}^m$ be arbitrary. Let $A = U\Sigma V^\top$ be an SVD of A with $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, and $V \in \mathbb{R}^{n\times n}$. Consider the least squares problem

$$\arg\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2. \tag{1}$$

a) Let $\mathbf{c} = U^\top \mathbf{b}$. Prove that $\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2 = \min_{\mathbf{y}\in\mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$.

b) Let $\sigma_1 \geq \cdots \geq \sigma_r$ denote the non-zero singular values of A (r is the rank of A). In particular, we have $\Sigma_{ii} = \sigma_i$ for all $i \in [r]$. Find a formula for the optimal solution $\mathbf{y}^* = \arg\min_{\mathbf{y}\in\mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$ in terms of $\sigma_1, \dots, \sigma_r$ and $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_m \end{bmatrix}^\top$.

c) Let $\mathbf{x}^*$ be the optimal solution $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2$. Given the optimal solution $\mathbf{y}^* = \arg\min_{\mathbf{y}\in\mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$ and the SVD of A, how can you compute $\mathbf{x}^*$?

5. Euclidean norm and 1-norm (Manhattan distance) (★★★)

This exercise includes Challenge 57. You can find the relevant definitions in Section 8.2 of the lecture notes. While Section 8.2 was not covered in the lecture, we still decided to include this exercise here since it does not require much extra theory (only the definition of the 1-norm) and we think it is a nice exercise.

a) Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary. Prove that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$.

Hint: Try to first prove it for $n = 2$. It might help to square both sides of the inequality first.

b) Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary. Prove that $\|\mathbf{x}\|_1 \leq \sqrt{n}\,\|\mathbf{x}\|_2$.

Hint: Recall the Cauchy-Schwarz inequality and try to find a nice way to write $\|\mathbf{x}\|_1$ as the dot product of two vectors.

6. Matrix norms (★★★)

This exercise includes Challenge 59. You can find the relevant definitions in Section 8.2 of the lecture notes. While Section 8.2 was not covered in the lecture, we still decided to include this exercise here since it does not require much extra theory (only the definition of the Frobenius norm and the operator norm) and we think it is a nice exercise.

Let $A \in \mathbb{R}^{m\times n}$ be arbitrary and let $\sigma_1 \geq \cdots \geq \sigma_{\min\{m,n\}}$ be its singular values.

a) Prove that $\|A\|_F^2 = \mathrm{Tr}(A^\top A)$.

b) Prove that $\|A\|_F^2 = \sum_{i=1}^{\min\{m,n\}} \sigma_i^2$.

c) Prove that $\|A\|_{op} = \sigma_1$.

d) Prove that $\|A\|_{op} \leq \|A\|_F$.

e) Prove that $\|A\|_F \leq \sqrt{\min\{m,n\}}\,\|A\|_{op}$.