D-INFK Linear Algebra HS 2024
Bernd Gärtner
Robert Weismantel

Assignment 2

Submission Deadline: **8 October, 2024** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$. Determine the rank of A. You need to justify your answer.

2. Rank-1 matrices (hand-in) (★☆☆)

Consider the 3×3 matrix

$$A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} = \mathbf{v}\mathbf{w}^\top$$

with $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$. Recall that by Lemma 2.21, A has rank 1 if and only if both $\mathbf{v}$ and $\mathbf{w}$ are non-zero. Otherwise, A has rank 0.

a) Assume now $v_1 \neq 0$ and $w_1 \neq 0$. Find a non-zero vector $\mathbf{x}$ (expressed in terms of $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$) that satisfies $A\mathbf{x} = \mathbf{0}$ (non-zero means that it cannot be the zero-vector $\mathbf{0}$).

b) Recall that we call a set of vectors $H \subseteq \mathbb{R}^m$ a hyperplane of $\mathbb{R}^m$ if and only if there exists a non-zero vector $\mathbf{d} \in \mathbb{R}^m$ such that $H = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$. We still assume that $v_1 \neq 0$ and $w_1 \neq 0$. Consider the set of vectors $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. Prove that $\mathcal{L}$ is a hyperplane of $\mathbb{R}^3$.

3. Matrix multiplication (★★☆)

a) For a natural number $k \geq 1$, we define the k-th power of a square matrix A as $A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$ where $\times$ denotes matrix multiplication. Moroever, we define $A^0 = I$.

Now consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find $x, y, z \in \mathbb{R}$ such that $A^3 + xA^2 + yA + zI = 0$. Note that both I and 0 are 3×3 matrices in this equation.

b) Let A and B be $m \times m$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

c) We say that a square matrix A is nilpotent if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The minimal $k \in \mathbb{N}$ such that $A^k = 0$ is called the nilpotent degree of A.

Let A be a nilpotent matrix of degree $k \in \mathbb{N}$, and B be a matrix commuting with A. In particular, note that both A and B are square matrices. Is AB nilpotent? If yes, what can we say about the nilpotent degree of AB?

d) Let A be an $m \times m$ nilpotent matrix of degree $k \in \mathbb{N}$. Prove that $(I - A)(I + A + \ldots + A^{k-1}) = I$.

e) Let T be an $m \times m$ upper triangular matrix. Assume that the diagonal of T consists of 0's only. Prove that $T^m = 0$, i.e. T is nilpotent of degree less or equal to m.

***Hint**: Even if you do not manage to solve a question, you can use its result to tackle subsequent questions.*

4. Scalar product (★★☆)

Recall that the scalar product of two vectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

in $\mathbb{R}^n$ is a real number given by

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \,.$$

and that vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

Let $A \in \mathbb{R}^{m \times n}$ be the matrix

$$A = \begin{bmatrix} — & \mathbf{u}_1 & — \\ — & \mathbf{u}_2 & — \\ & \vdots & \\ — & \mathbf{u}_m & — \end{bmatrix}$$

with rows $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in \mathbb{R}^n$. Recall that, by Observation 2.7, we have

$$A\mathbf{x} = \begin{bmatrix} — & \mathbf{u}_1 & — \\ — & \mathbf{u}_2 & — \\ & \vdots & \\ — & \mathbf{u}_m & — \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{x} \\ \mathbf{u}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{u}_m \cdot \mathbf{x} \end{bmatrix}$$

for $\mathbf{x} \in \mathbb{R}^n$. In particular, we have $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x}$ is perpendicular to each of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$.

a) Now consider two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfying $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ and let $\lambda, \mu \in \mathbb{R}$ be arbitrary. Prove that the vector $\lambda\mathbf{x} + \mu\mathbf{y}$ is perpendicular to each of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$.

b) Finally, consider the set of vectors $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ and assume $|\mathcal{L}| \geq 2$. Is $\mathcal{L}$ a finite set?

5. Linear independence (★☆☆)

a) What is the rank of the following 2×3 matrix A?

$$A = \begin{bmatrix} 1 & -3 & 3 \\ -2 & 6 & 0 \end{bmatrix}$$

b) What is the rank of the following 3×3 matrix A? You may use the (yet unproven) statement from the lecture that says that one can choose any order on the columns of a matrix to compute its rank.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

6. Transpose (★☆☆)

a) Find two 2×2 matrices A and B such that $(AB)^\top \neq A^\top B^\top$.

b) Can you also find two symmetric 2×2 matrices A, B with $(AB)^\top \neq A^\top B^\top$?

7. Multiple choice Let A be an $m_1 \times n_1$ matrix and let B be an $m_2 \times n_2$ matrix for natural numbers m_1, n_1, m_2, n_2. For each statement, determine whether it is true or not (regardless of what values m_1, n_1, m_2, n_2 take).

1. If A^2 is defined, then A must be square.

(**a**) Yes

(**b**) No

2. If $A^2 = I$, then $A = I$.

(**a**) Yes

(**b**) No

3. If $A^3 = 0$, then $A = 0$.

(**a**) Yes

(**b**) No

4. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$ for all $n \in \mathbb{N}$.

(**a**) Yes

(**b**) No

5. If $AB = B$ for some choice of B, then $A = I$.

(**a**) Yes

(**b**) No

6. If both products AB and BA are defined, then A and B must be square.

(**a**) Yes

(**b**) No

7. If both products AB and BA are defined, then AB and BA must be square.

(**a**) Yes

(**b**) No

8. If two columns of A are equal and AB is defined, the corresponding columns of AB must also be equal.

(**a**) Yes

(**b**) No

9. If two columns of B are equal and AB is defined, the corresponding columns of AB must also be equal.

(**a**) Yes

(**b**) No

10. If two rows of A are equal and AB is defined, the corresponding rows of AB must also be equal.

(**a**) Yes

(**b**) No

11. If two rows of B are equal and AB is defined, the corresponding rows of AB must also be equal.

(**a**) Yes

(**b**) No

12. If A and B are symmetric matrices and AB is defined, AB is also symmetric.

(**a**) Yes

(**b**) No