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Assignment 3

Submission Deadline: **15 October, 2024** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get feedback and bonus points for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. **Linear transformations (in-class)** (★☆☆)

   a) Let $n \in \mathbb{N}^+$. Consider the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by

   $$T(\mathbf{x}) := \sum_{k=1}^{n} k x_k$$

   for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$. Prove that T is a linear transformation.

   b) Let $n \in \mathbb{N}^+$ with $n \geq 2$ be arbitrary. Consider the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by

   $$T(\mathbf{x}) := \sum_{k=1}^{n} (x_k)^k$$

   for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$. Is T a linear transformation?

2. **Linear transformation (bonus, hand-in)** (★☆☆)

   Let $m, n \in \mathbb{N}^+$ and consider an arbitrary $m \times (n+1)$ matrix

   $$A = \begin{bmatrix} | & | & & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{v}_{n+1} \\ | & | & & | & | \end{bmatrix}$$

   with columns $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{R}^m$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the function defined by

   $$T(\mathbf{x}) := A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix}$$

   for all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Prove that T is a linear transformation if and only if $\mathbf{v}_{n+1} = \mathbf{0}$.

3. Rotation matrices (★★☆)

Hint: *This exercise requires some basic knowledge of* sin *and* cos. *Part c) can also be solved independently by assuming parts a) and b).*

a) We define a real 2×2 matrix A to be a *rotation matrix* if and only if there exists a rotation angle $\phi \in \mathbb{R}$ such that

$$A = Q(\phi) := \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Prove that the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is a rotation matrix according to this definition.

b) Show that the matrix product $Q(\phi_1)\,Q(\phi_2)$ of two rotation matrices with angles ϕ_1 and ϕ_2 is again a rotation matrix $Q(\phi_3)$ according to this definition, and determine the corresponding rotation angle ϕ_3.

Hint: *You might need to review trigonometric formulas to solve this question.*

c) Let A be a 2×2 rotation matrix. Prove that there exists a 2×2 matrix B such that $AB = BA = I$.

4. Product of triangular matrices (★★☆)

a) Let A and B be $m \times m$ lower triangular matrices. Prove that AB is lower triangular.

b) Let A and B be $m \times m$ upper triangular matrices. Prove that AB is upper triangular.

Hint: *Use the statement from subtask a).*

5. Linear transformations (★★☆)

a) Consider the linear transformation given by the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Describe the geometric operation that this linear transformation corresponds to.

b) Consider the line $L = \{\lambda\mathbf{v} : \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^3$ with $\mathbf{v} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$. Find a matrix $A \in \mathbb{R}^{3\times 3}$ that corresponds to rotating vectors by $180°$ around the axis L.

Hint: *Try to find out what $A\mathbf{e}_1$, $A\mathbf{e}_2$, and $A\mathbf{e}_3$ should be.*

6. Reconstruct a linear transformation (★☆☆)

a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},\ T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

Determine the general formula for $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ with $x, y \in \mathbb{R}$.

b) Find a matrix A such that $T_A = T$.

7. Embedding a line in R^m (★☆☆)

Let $\mathbf{v} \in \mathbb{R}^m$ be non-zero and consider the line $L = \{\lambda\mathbf{v} \;:\; \lambda \in \mathbb{R}\}$. Prove that there is a linear transformation $T : \mathbb{R} \to \mathbb{R}^m$ with $\{T(\mathbf{x}) \;:\; \mathbf{x} \in \mathbb{R}\} = L$ (i.e. the image of T is exactly L).