D-INFK
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Linear Algebra

HS 2024

Assignment 6

Submission Deadline: **05 November, 2024** at 23:59
Course Website: `https://ti.inf.ethz.ch/ew/courses/LA24/index.html`

Exercises

You can get feedback from your TA and bonus points for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. **Subspaces of vector spaces (in-class)** (★★☆)

   a) Let H be a hyperplane of $\mathbb{R}^m$. Recall that this means that there exists a non-zero vector $\mathbf{d} \in \mathbb{R}^m$ with $H = \{\mathbf{v} \cdot \mathbf{d} = 0 : \mathbf{v} \in \mathbb{R}^m\}$. Prove that H is a subspace of $\mathbb{R}^m$.

   b) Consider again a hyperplane H of $\mathbb{R}^m$. Prove that the dimension of H is $m - 1$.

   c) In this exercise we consider the vector space V of all real-valued functions on the interval $[0, 1]$. In other words, every element $\mathbf{f} \in V$ is a function $\mathbf{f} : [0, 1] \to \mathbb{R}$ and conversely, every function $\mathbf{f} : [0, 1] \to \mathbb{R}$ is in V. Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition $\mathbf{f} + \mathbf{g}$ of such functions $\mathbf{f} \in V$ and $\mathbf{g} \in V$, and a valid scalar multiplication $c\mathbf{f}$ for a scalar $c \in \mathbb{R}$ and $\mathbf{f} \in V$ defined as follows:

   $$(\mathbf{f} + \mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in [0, 1]$$
   $$(c\mathbf{f})(x) := c\mathbf{f}(x) \qquad \text{for all } \mathbf{f} \in V, x \in [0, 1] \text{ and } c \in \mathbb{R}.$$

   Prove that
   $$U = \{\mathbf{f} \in V : \mathbf{f}(x) = \mathbf{f}(1 - x) \text{ for all } x \in [0, 1]\} \subseteq V$$
   is a subspace of V.

2. **Subspace of matrices (bonus, hand-in)** (★★☆)

   Let $m \in \mathbb{N}^+$. Fix an arbitrary non-zero vector $\mathbf{v} \in \mathbb{R}^m$ and consider the set of matrices $S^{\mathbf{v}} := \{A \in \mathbb{R}^{2\times m} : A\mathbf{v} = \mathbf{0}\} \subseteq \mathbb{R}^{2\times m}$. It is not hard to show that $S^{\mathbf{v}}$ is a subspace of $\mathbb{R}^{2\times m}$ (you do not have to show this, you can assume it without proof). What is the dimension of $S^{\mathbf{v}}$? Prove your answer.

   Hint*: You can use the statements from exercise 1a) and 1b) without proof, even if you did not solve them.*

3. **Subspaces** (★★☆)

   Let V be a vector space and let U and W be subspaces of V. Show that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

4. **Symmetric matrices** (★★☆)

Let $m \in \mathbb{N}^+$ be arbitrary. Consider the set of symmetric $m \times m$ matrices $\mathcal{S}_m$ which is a subspace of $\mathbb{R}^{m \times m}$. What is the dimension of $\mathcal{S}_m$?

5. **Odd and even functions** (★★★)

In this exercise, we consider the vector space V of all real-valued functions on $\mathbb{R}$. In other words, every element $\mathbf{f} \in V$ is a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ and conversely, every function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is in V. Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition $\mathbf{f} + \mathbf{g}$ of such functions $\mathbf{f} \in V$ and $\mathbf{g} \in V$, and a valid scalar multiplication $c\mathbf{f}$ for a scalar $c \in \mathbb{R}$ and $\mathbf{f} \in V$ defined as follows:

$$(\mathbf{f} + \mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in \mathbb{R}$$
$$(c\mathbf{f})(x) := c\mathbf{f}(x) \qquad \text{for all } \mathbf{f} \in V, x \in \mathbb{R} \text{ and } c \in \mathbb{R}.$$

Now consider the set of odd functions

$$O = \{\mathbf{f} \in V : \mathbf{f}(-x) = -\mathbf{f}(x) \text{ for all } x \in \mathbb{R}\}$$

and the set of even functions

$$E = \{\mathbf{f} \in V : \mathbf{f}(-x) = \mathbf{f}(x) \text{ for all } x \in \mathbb{R}\}.$$

a) Prove that both O and E are subspaces of V.

b) Prove that the intersection $O \cap E$ contains only the zero function $\mathbf{0} : \mathbb{R} \to \mathbb{R}$ with $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

c) Prove that any function $\mathbf{f} \in V$ can be written as $\mathbf{f} = \mathbf{g} + \mathbf{h}$ for some $\mathbf{g} \in E$ and $\mathbf{h} \in O$.

6. **Basis of subspace of polynomials** (★☆☆)

Consider the three polynomials $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}[x]$ defined as

$$\mathbf{p} = x^3 + x, \quad \mathbf{q} = x^2 + 1, \quad \mathbf{r} = x^2 + x + 1.$$

What is the dimension of $\mathsf{Span}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \subseteq \mathbb{R}[x]$? Prove your answer.

7. **1.** Let U_1, U_2 be arbitrary subspaces of $\mathbb{R}^m$. Which of the following subsets of $\mathbb{R}^m$ must be subspaces of $\mathbb{R}^m$ as well?

(a) $U_1 \cap U_2$

(b) $U_1 \cup U_2$

(c) $U_1 \setminus U_2 := \{\mathbf{u} \in U_1 : \mathbf{u} \notin U_2\}$

(d) $\varnothing$

(e) $\{\mathbf{0}\}$

(f) $U_1 + U_2 := \{\mathbf{u}_1 + \mathbf{u}_2 : \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2\}$

2. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

Which of the following sets of vectors is a basis of $\mathbb{R}^4$?

(a)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(c)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

3. Which of the following matrices are in row echelon form?

(**a**) $$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(**b**) $$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(**c**) $$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(**d**) $$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$