D-INFK
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Linear Algebra

HS 2024

Assignment 7

Submission Deadline: **12 November, 2024** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Solving linear systems (in-class) (★☆☆)

Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} -1 & 2 & 5 & -2 \\ -3 & 3 & 12 & -3 \\ 1 & -14 & -7 & -6 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} -6 \\ -15 \\ 8 \end{pmatrix}.$$

a) Determine the set of solutions $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{b}\}$, i.e. write down an *explicit* characterization of this set of solutions.

Hint: In the lecture you learned that every solution can be obtained from a particular solution and a basis of $\mathbf{N}(A)$. Hence, an explicit characterization of $\mathcal{L}$ can be given by finding such a particular solution and a basis of $\mathbf{N}(A)$ and then describing the possible combinations that are solutions.

b) Write down a basis for $\mathbf{N}(A)$ (you might have already found it in the previous subtask), and also find a basis for $\mathbf{C}(A)$.

c) What are the dimensions of $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A^\top)$, and $\mathbf{C}(A^\top)$?

d) Determine a basis of $\mathbf{C}(A^\top)$.

2. Nullspace and column space (hand-in) (★★☆)

Let $\mathbf{v}$ be a *unit vector* (i.e. $\| \mathbf{v} \| = 1$) in $\mathbb{R}^3$. Consider the 3×3 matrices A and P defined by

$$A := \mathbf{v}\mathbf{v}^\top, \qquad P := I_3 - \mathbf{v}\mathbf{v}^\top = I_3 - A$$

where I_3 is the 3×3 identity matrix.

a) What can you say about the relations between A^2 and A, and P^2 and P?

b) Let $\mathbf{w} \in \mathbb{R}^3$ be orthogonal to $\mathbf{v}$ (i.e. $\mathbf{w} \cdot \mathbf{v} = 0$). Prove $A\mathbf{w} = \mathbf{0}$.

c) Now let $\mathbf{w} \in \mathbb{R}^3$ be a vector satisfying $A\mathbf{w} = \mathbf{0}$. Prove $\mathbf{w} \cdot \mathbf{v} = 0$.

d) Based on b) and c), describe the nullspace $\mathbf{N}(A)$.

e) Determine the rank of A. Is A invertible?

f) Prove that $\mathbf{C}(A) = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$.

g) Also prove that $\mathbf{C}(A) = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$.

h) Use g) to prove $\mathbf{N}(P) = \mathbf{C}(A)$.

i) Finally, prove $\mathbf{C}(P) = \mathbf{N}(A)$.

Hint: In every subtask you may of course use statements that you have already proven in previous subtasks. For some of the subtasks we specifically tell you which previous subtasks might be helpful.

3. Reconstruct a matrix (★☆☆)

Let A be a 3×2 matrix satisfying

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

a) Determine A.

b) Determine the dimensions of the four fundamental subspaces $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A^\top)$, $\mathbf{C}(A^\top)$ of A.

4. Dimension of a matrix subspace (★★☆)

Consider the subspace of matrices with trace 0 defined as

$$S := \{A \in \mathbb{R}^{m \times m} : \mathsf{Tr}(A) = 0\},$$

where $\mathsf{Tr}(A)$ denotes the sum of the diagonal elements of A. As stated in the lecture, S is a subspace of $\mathbb{R}^{m \times m}$ (you do not need to prove this). Determine the dimension of S.

5. Convex combinations in two dimensions (★★☆)

For this exercise, we will need the notion of a line segment. Consider an arbitrary set $S \subseteq \mathbb{R}^2$. We say that S is a line segment if and only if there exist two distinct points $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ such that

$$S = \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 : \lambda_1, \lambda_2 \in \mathbb{R}_0^+, \lambda_1 + \lambda_2 = 1\}.$$

In other words, S is the set of convex combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ (see lecture notes Section 1.1.4). Try to convince yourself that this characterization of line segments corresponds to what you intuitively think of as a line segment. It might help to draw some examples.

We will also need a more concrete notion of a triangle: We say that a set $S \subseteq \mathbb{R}^2$ is a triangle if and only if S is not a line segment and there exist three distinct points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ such that

$$S = \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_0^+, \lambda_1 + \lambda_2 + \lambda_3 = 1\}.$$

Again, convince yourself that this characterization intuitively makes sense by drawing some examples.

Let now $S = \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_0^+, \lambda_1 + \lambda_2 + \lambda_3 = 1\}$ for some arbitrary (not necessarily distinct) vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$, and consider the 2×2 matrix A defined as

$$A = \begin{bmatrix} | & | \\ (\mathbf{v}_2 - \mathbf{v}_1) & (\mathbf{v}_3 - \mathbf{v}_1) \\ | & | \end{bmatrix}.$$

a) Prove that S is either a single point, a triangle, or a line segment.

b) Prove that A has rank 0 if and only if S is a point.

c) Prove that A has rank 2 if and only if S is a triangle.

d) Prove that A has rank 1 if and only if S is a line segment.

Hint: *Use previous subtasks.*

6. 1. Which of the following statements is true for all $m \times m$ matrices A?

(**a**) $\mathbf{N}(A) = \mathbf{N}(2A)$

(**b**) $\mathbf{N}(A) = \mathbf{N}(A^2)$

(**c**) $\mathbf{N}(A) = \mathbf{N}(A + I)$

(**d**) $\mathbf{N}(A) = \mathbf{N}(A^\top)$

2. Which of the following statements is true for all square matrices A?

(**a**) $\mathbf{C}(A) = \mathbf{C}(2A)$

(**b**) $\mathbf{C}(A) = \mathbf{C}(A^2)$

(**c**) $\mathbf{C}(A) = \mathbf{C}(A + I)$

(**d**) $\mathbf{C}(A) = \mathbf{C}(A^\top)$

3. The following equations each describe a plane in $\mathbb{R}^3$:

$$\begin{array}{rcrcrcl} x & - & y & - & z & = & 0 \\ 2x & - & 5y & + & 3z & = & 0 \\ 3x & & & + & 4z & = & 0. \end{array}$$

Which of the following statements is true?

(**a**) The intersection of all three planes is empty.

(**b**) The intersection of all three planes contains exactly one element.

(**c**) The intersection of all three planes is a line.

4. Consider the linear system

$$x_1 + (b-1)x_2 = 3$$
$$-3x_1 - (2b-8)x_2 = -5$$

with variables x_1, x_2 and parameter $b \in \mathbb{R}$. For which values of b is the set of solutions to the above system empty (i.e. there is no solution)?

(**a**) Only for $b = 0$.

(**b**) Only for $b = -5$.

(**c**) For all possible values of b (i.e. for all of $\mathbb{R}$).

(**d**) The system always has a solution regardless of the value of b.