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Solution for Assignment 0

1. Linear combinations of vectors (★☆☆)

a) Consider an arbitrary vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$. We define $\lambda := \frac{b_1+b_2}{2}$ and $\mu := \frac{b_2-b_1}{2}$. Then we have

$$\lambda \mathbf{v} + \mu \mathbf{w} = \begin{bmatrix} \lambda - \mu \\ \lambda + \mu \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathbf{b} \tag{1}$$

which proves that $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. While this proves the claim and is considered a complete solution, let us also explain how to find out that we should pick $\lambda := \frac{b_1+b_2}{2}$ and $\mu := \frac{b_2-b_1}{2}$. For this, assume that we do not know yet what values λ and μ should have. We can still write down Equation 1, interpreting λ and μ as unknowns. In fact, this gives us a system of two equations

$$\begin{aligned} \lambda - \mu &= b_1 \\ \lambda + \mu &= b_2, \end{aligned}$$

one for each coordinate. Adding the two equations we obtain $2\lambda = b_1 + b_2$ and hence $\lambda = \frac{b_1+b_2}{2}$. On the other hand, subtracting the first equation from the second gives $2\mu = b_2 - b_1$ and hence $\mu = \frac{b_2-b_1}{2}$. One can also think about this geometrically (see Figure 1), analogous to the "row picture" from Section 1.1.3 in the lecture notes.

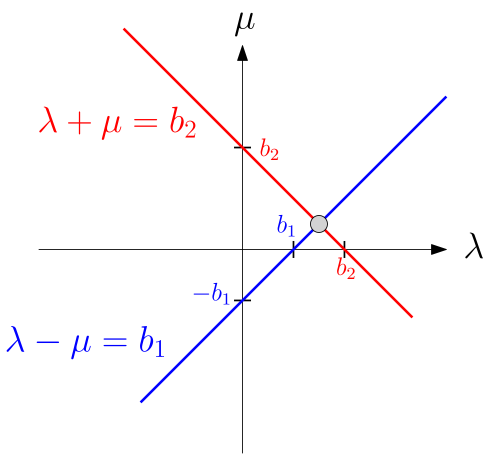

Figure 1: The lines given by the two equations intersect at $(\frac{b_1+b_2}{2}, \frac{b_2-b_1}{2})$.

b) Consider the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$. In order to write it as a linear combination of $\mathbf{v}$ and $\mathbf{w}$, we would have to find $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda \mathbf{v} + \mu \mathbf{w} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{b}.$$

Each coordinate provides a constraint on the values of λ and μ. In particular, from the first coordinate we get the constraint $\lambda = 0$, from the second coordinate we get the constraint $\mu = 1$, and from the third coordinate we get $\lambda + \mu = 0$. This system of three equations in two unknowns does not have a solution and hence we conclude that $\mathbf{b}$ cannot be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

2. The perfect long drink (★☆☆)

a) Mixing $\frac{3}{5}$ of the first imperfect drink with $\frac{2}{5}$ of the second imperfect drink will yield a perfect drink. To see this, represent the imperfect drinks as vectors $\begin{bmatrix}15\\85\end{bmatrix} \in \mathbb{R}^2$ and $\begin{bmatrix}35\\65\end{bmatrix} \in \mathbb{R}^2$, and the perfect drink as $\begin{bmatrix}23\\77\end{bmatrix} \in \mathbb{R}^2$. It remains to check that

$$\frac{3}{5}\begin{bmatrix}15\\85\end{bmatrix} + \frac{2}{5}\begin{bmatrix}35\\65\end{bmatrix} = \begin{bmatrix}23\\77\end{bmatrix}.$$

While this is already a full solution, we describe again how one could arrive at $\frac{3}{5}$ and $\frac{2}{5}$, respectively. We want to find λ and μ such that

$$\lambda\begin{bmatrix}15\\85\end{bmatrix} + \mu\begin{bmatrix}35\\65\end{bmatrix} = \begin{bmatrix}23\\77\end{bmatrix}.$$

This gives us the two equations

$$\begin{aligned}15\lambda + 35\mu &= 23\\ 85\lambda + 65\mu &= 77\end{aligned}$$

which can be solved in various ways. One way is to add the two equations to get $100\lambda + 100\mu = 100$ and thus $\lambda + \mu = 1$. Now plug in $\lambda = 1 - \mu$ into the first equation to get $\mu = \frac{2}{5}$ and hence $\lambda = 1 - \mu = \frac{3}{5}$.

b) The set $\hat{D}$ can be written down as

$$\hat{D} = \{\lambda\mathbf{v} + \mu\mathbf{w} \in \mathbb{R}^2 : \lambda \geq 0, \mu \geq 0, \lambda + \mu = 1\}.$$

In words, $\hat{D}$ consists of all vectors that can be written as linear combinations $\lambda\mathbf{v} + \mu\mathbf{w}$ that additionally satisfy $\lambda \geq 0$, $\mu \geq 0$ and $\lambda + \mu = 1$. We need the constraints $\lambda \geq 0$ and $\mu \geq 0$ because we cannot use a negative amount of one of the drinks. Moreover, we need to make sure that we are mixing a drink of size 100ml, hence the constraint $\lambda + \mu = 1$. In math lingo, $\hat{D}$ is the set of convex combinations of $\mathbf{v}$ and $\mathbf{w}$ (see Definition 1.7 in the lecture notes). Geometrically, $\hat{D}$ has the shape of a line segment that connects the points $\mathbf{v}$ and $\mathbf{w}$ (see also Figure 1.10 in the lecture notes).

c) Since the mixed drink does not need to contain exactly 100ml anymore, we can get rid of the constraint $\lambda + \mu = 1$. But we still cannot use $\mathbf{v}$ and $\mathbf{w}$ more than once each, i.e. we need to include the constraints $\lambda \leq 1$ and $\mu \leq 1$ instead. We get the parallelogram (see Figure 2)

$$\overline{D} = \{\lambda\mathbf{v} + \mu\mathbf{w} \in \mathbb{R}^2 : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\}.$$

3. Geometry of linear combinations (★☆☆)

This example was taken from the lecture notes (Figure 1.22). The set in a) has the shape of a line, the set in b) the shape of a plane, and the set in c) is all of $\mathbb{R}^3$.

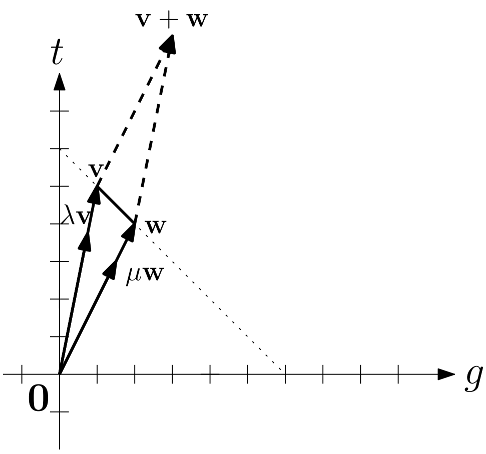

Figure 2: A sketch of the situation with the two imperfect drinks $\mathbf{v}$ and $\mathbf{w}$. The set D is shown as a dotted line between the two axis. The set $\hat{D}$ is shown in bold connecting $\mathbf{v}$ and $\mathbf{w}$. The set $\overline{D}$ is the parallelogram. We can see in the picture that any linear combination $\lambda\mathbf{v} + \mu\mathbf{w}$ with $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$ must land inside the parallelogram. Moreover, any point inside the parallelogram can be reached by choosing λ and μ appropriately. The argument for this is given in the "column picture" from Section 1.1.3 of the lecture notes.

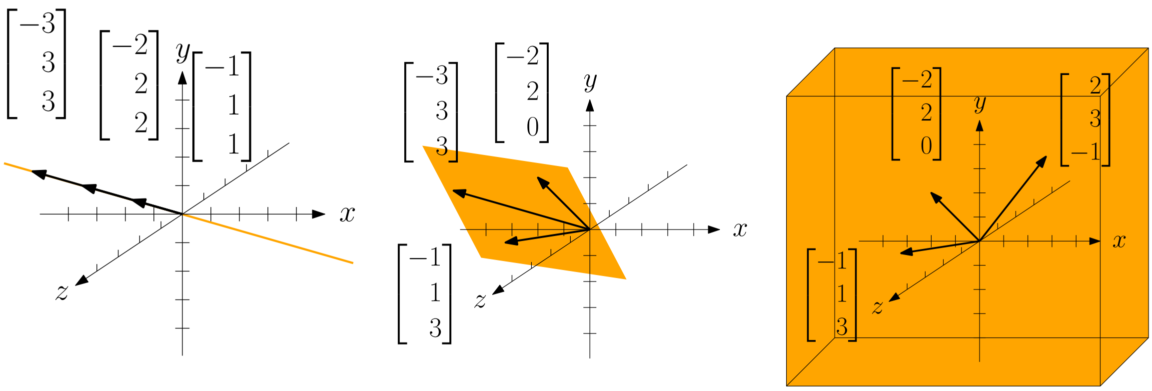

Figure 3: Solutions to 3a, 3b, and 3c, respectively.