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Solution for Assignment 1

1. **a)** By definition of L, there exists $\mathbf{w} \in \mathbb{R}^m$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. In particular, we can write $\mathbf{u} = \lambda_u \mathbf{w}$ for some $\lambda_u \in \mathbb{R}$. Observe that we have $\lambda_u \neq 0$ because $\mathbf{u} \neq \mathbf{0}$. Now consider an arbitrary vector $\mathbf{v} \in L$. There must be $\lambda_v \in \mathbb{R}$ such that $\mathbf{v} = \lambda_v \mathbf{w}$. Putting these together, we get $\mathbf{v} = \frac{\lambda_v}{\lambda_u}\mathbf{u}$. This already proves $L \subseteq \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$ and it remains to prove $\{\lambda \mathbf{u} : \lambda \in \mathbb{R}\} \subseteq L$. For this, consider an arbitrary vector $\mathbf{v}' = \lambda_{v'} \mathbf{u} \in \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$. Combining $\mathbf{v}' = \lambda_{v'}\mathbf{u}$ with $\mathbf{u} = \lambda_u \mathbf{w}$ we get $\mathbf{v}' = \lambda_{v'}\lambda_u \mathbf{w}$ and hence $\mathbf{v}' \in L$. We conclude that $L = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$.

b) Let L_1 and L_2 be two lines of $\mathbb{R}^m$. By definition, there exist $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $L_1 = \{\lambda \mathbf{w}_1 : \lambda \in \mathbb{R}\}$ and $L_2 = \{\lambda \mathbf{w}_2 : \lambda \in \mathbb{R}\}$. In order to see that $\mathbf{0} \in L_1 \cap L_2$, it suffices to observe $\mathbf{0} \in L_1$ and $\mathbf{0} \in L_2$ since we have $\mathbf{0} = 0\mathbf{w}_1 = 0\mathbf{w}_2$. Now assume $L_1 \cap L_2 \neq \{\mathbf{0}\}$. Because $L_1 \cap L_2$ is not empty, there exists a non-zero vector $\mathbf{u} \in L_1 \cap L_2$. By $\mathbf{u} \in L_1$, we know from part a) that $L_1 = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$. Analogously, we have $L_2 = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}$ and hence $L_1 = L_2$.

2. **a)** By definition, there must be a vector $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$ such that $L = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\}$. We want to find a vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathbb{R}^2$ such that $L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$. In particular, we want $\mathbf{w} \cdot \mathbf{d} = w_1 d_1 + w_2 d_2 \overset{!}{=} 0$ since $\mathbf{w} \in L$. Choosing $d_1 := -w_2$ and $d_2 := w_1$ would certainly work, so let this be our "guess" for $\mathbf{d}$. It remains to prove that with this choice of $\mathbf{d}$, we have $L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$.

$\subseteq$: Consider an arbitrary element $\mathbf{u} = \lambda_u \mathbf{w} \in L$. We have

$$\mathbf{u} \cdot \mathbf{d} = (\lambda_u \mathbf{w}) \cdot \mathbf{d} = \lambda_u w_1 d_1 + \lambda_u w_2 d_2 = \lambda_u (w_1 d_1 + w_2 d_2) = 0$$

and hence $\mathbf{u} \in \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$.

$\supseteq$: Consider an arbitrary element $\mathbf{v} \in \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$. In particular, we have $v_1 d_1 + v_2 d_2 = -v_1 w_2 + v_2 w_1 = 0$. Our goal is to find λ such that $\mathbf{v} = \lambda \mathbf{w}$. Recall from the definition of a line that we must have $\mathbf{w} \neq \mathbf{0}$ and hence either $w_1 \neq 0$ or $w_2 \neq 0$. Assume first $w_1 \neq 0$ and observe that we can rewrite $-v_1 w_2 + v_2 w_1 = 0$ to $v_2 = \frac{w_2}{w_1} v_1$. Choosing $\lambda = \frac{v_1}{w_1}$ we can see that indeed, we have $v_1 = \lambda w_1$ and $v_2 = \lambda w_2$, as desired. If we have $w_1 = 0$, then it must be the case that $w_2 \neq 0$. But then we can rewrite $-v_1 w_2 + v_2 w_1 = 0$ to $v_1 = \frac{w_1}{w_2} v_2$ and choose $\lambda = \frac{v_2}{w_2}$.

b) In order to prove that S' is a subset of a hyperplane of $\mathbb{R}^{m+1}$, we need to find a non-zero vector $\mathbf{d}' \in \mathbb{R}^{m+1}$ such that $\mathbf{w} \cdot \mathbf{d}' = 0$ for all $\mathbf{w} \in S'$. To achieve this, consider what we know about an arbitrary $\mathbf{w} \in S'$: It must be of the form

$$\mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \\ 1 \end{bmatrix}$$

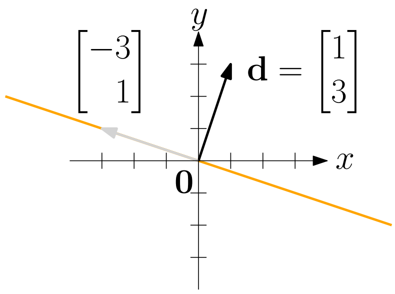

Figure 1: This figure illustrates the situation in task a). The yellow line given by $\{\lambda \begin{bmatrix}-3\\1\end{bmatrix} : \lambda \in \mathbb{R}\}$ is equal to the hyperplane $\{\mathbf{v} : \mathbf{v} \cdot \begin{bmatrix}1\\3\end{bmatrix} = 0\}$.

for some $\mathbf{v} \in S$. So for the scalar product $\mathbf{w} \cdot \mathbf{d}'$ we get

$$\mathbf{w} \cdot \mathbf{d}' = w_1 d'_1 + w_2 d'_2 + \cdots + w_{m+1} d'_{m+1} = v_1 d'_1 + v_2 d'_2 + \cdots + v_m d'_m + d'_{m+1}.$$

By $\mathbf{v} \in S$ we know that $\mathbf{v} \cdot \mathbf{d} = c$. So if we now choose

$$\mathbf{d}' = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \\ -c \end{bmatrix}$$

we get

$$\mathbf{w} \cdot \mathbf{d}' = v_1 d'_1 + v_2 d'_2 + \cdots + v_m d'_m + d'_{m+1} = (\mathbf{v} \cdot \mathbf{d}) - c = (\mathbf{v} \cdot \mathbf{d} - c) = 0$$

as desired. This works for any $\mathbf{w} \in S'$ and hence S' is a subset of the hyperplane $\{\mathbf{v} \in \mathbb{R}^{m+1} : \mathbf{v} \cdot \mathbf{d}'\}$.

3. **a)** Let $\mathbf{1} \in \mathbb{R}^m$ denote the vector whose entries are all 1. Now observe that

$$\sum_{i=1}^{m} v_i = \sum_{i=1}^{m} 1 v_i = \mathbf{1} \cdot \mathbf{v} \leq \|\,\mathbf{1}\,\|\,\|\,\mathbf{v}\,\|$$

where we used Cauchy-Schwarz in the inequality. It remains to observe that $\|\,\mathbf{1}\,\| = \sqrt{m}$.

b) Let $\mathbf{w} \in \mathbb{R}^m$ denote the vector whose i-th entry is $\sqrt{i}$ for all $i \in [m] := \{1, \ldots, m\}$. Observe that

$$\sum_{i=1}^{m} \sqrt{i} v_i = \mathbf{w} \cdot \mathbf{v} \leq \|\,\mathbf{w}\,\|\,\|\,\mathbf{v}\,\|$$

where we used Cauchy-Schwarz in the inequality. For the length $\|\,\mathbf{w}\,\|$ of $\mathbf{w}$, we get

$$\|\,\mathbf{w}\,\| = \sqrt{\sum_{i=1}^{m} (\sqrt{i})^2} = \sqrt{\sum_{i=1}^{m} i} = \sqrt{\frac{(m+1)m}{2}} = \sqrt{\frac{m^2}{2} + \frac{m}{2}} \leq \sqrt{\frac{m^2}{2} + \frac{m^2}{2}} = m.$$

4. **a)** This set of vectors is not linearly independent as $\mathbf{v}$ can always be written as a linear combination of any other vector, e.g. $0\mathbf{u} = \mathbf{v}$.

b) This set of vectors is linearly independent. To see this, we first check that $\mathbf{v}$ cannot be obtained as a linear combination of $\mathbf{u}$: indeed, observe that we cannot obtain the 1 in the third and fourth coordinate of $\mathbf{v}$ from $\mathbf{u}$ as $\mathbf{u}$ contains 0 in both of those coordinates.

Next, we check that $\mathbf{w}$ cannot be obtained as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Such a linear combination would require scalars λ and μ such that

$$\lambda\mathbf{u} + \mu\mathbf{v} = \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{w}.$$

But notice that the 1 in the second coordinate of $\mathbf{w}$ can only be obtained by setting $\lambda = 1$. Similarly, the 1 in the third coordinate of $\mathbf{w}$ can only be obtained with $\mu = 1$. But with this choice of λ and μ, we would get 1 (instead of 0) in the first and forth coordinate. So we conclude that there is no such linear combination.

From the lecture we know that checking the three vectors in any order (we did it in the order they were given) suffices to check linear independence. Hence, we conclude that the three vectors are linearly independent.

5. **a)** With $\lambda_i := (-1)^i$ for all $i \in \{1, \dots, m\}$, we get

$$\sum_{i=1}^{m} \lambda_i \mathbf{v}_i = \sum_{i=1}^{m-1} (-1)^i (\mathbf{e}_i + \mathbf{e}_{i+1}) + (-1)^m (\mathbf{e}_m + \mathbf{e}_1).$$

Now observe that we have

$$\sum_{i=1}^{m-1} (-1)^i (\mathbf{e}_i + \mathbf{e}_{i+1}) = -(\mathbf{e}_1 + \mathbf{e}_2) + (\mathbf{e}_2 + \mathbf{e}_3) - \cdots - (\mathbf{e}_{m-1} + \mathbf{e}_m)$$

which reduces to $-\mathbf{e}_1 - \mathbf{e}_m$. Plugging this into the previous equation, we therefore obtain

$$\sum_{i=1}^{m} \lambda_i \mathbf{v}_i = -\mathbf{e}_1 - \mathbf{e}_m + (-1)^m (\mathbf{e}_m + \mathbf{e}_1) = -\mathbf{e}_1 - \mathbf{e}_m + (\mathbf{e}_m + \mathbf{e}_1) = \mathbf{0}.$$

By Lemma 1.19, we conclude that the vectors are linearly dependent.

b) Consider arbitrary $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $\lambda_1\mathbf{v}_1 + \cdots + \lambda_m\mathbf{v}_m = \mathbf{0}$. By definition of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$, the i-th coordinate is non-zero only in $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ for all $i \in \{2, \dots, m\}$, and the first coordinate is non-zero only in $\mathbf{v}_1$ and $\mathbf{v}_m$. Moreover, every non-zero entry is exactly 1. Hence, we must have $\lambda_i = -\lambda_{i-1}$ for all $i \in \{2, \dots, m\}$, and $\lambda_1 = -\lambda_m$. Therefore, we get

$$\lambda_1 = -\lambda_2 = -(-\lambda_3) = \cdots = (-1)^{m-1}\lambda_m = (-1)^m \lambda_1,$$

which implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ since m is odd. By Lemma 1.19, we conclude that the vectors are linearly independent.

6. We first observe that we have $\| \mathbf{v} \| = \sqrt{x^2 + y^2 + z^2} = \sqrt{z^2 + x^2 + y^2} = \| \mathbf{w} \|$. In particular, this implies $\| \mathbf{v} \| \| \mathbf{w} \| = x^2 + y^2 + z^2$. Using the formula from the lecture for the angle α between $\mathbf{v}$ and $\mathbf{w}$, we calculate

$$\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{v} \| \| \mathbf{w} \|} = \frac{xz + yx + zy}{x^2 + y^2 + z^2}.$$

Next, observe that we can rewrite $xz + yx + zy = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$. By our assumption $x + y + z = 0$, the first term vanishes and we obtain

$$\cos(\alpha) = \frac{xz + yx + zy}{x^2+y^2+z^2} = \frac{-\frac{1}{2}(x^2+y^2+z^2)}{x^2+y^2+z^2} = -\frac{1}{2}.$$

To find α, it remains to look up (or remember) $\cos^{-1}(-\frac{1}{2}) = \frac{2}{3}\pi$ ($= 120°$).

7. We follow roughly the proof of Fact 1.5 and write down the two equations

$$\begin{aligned} v_1\lambda + w_1\mu &= u_1 \\ v_2\lambda + w_2\mu &= u_2 \end{aligned}$$

that we want to solve for $\lambda, \mu \in \mathbb{R}$. Since we assumed $\mathbf{v} \neq \mathbf{0}$, we know that either $v_1 \neq 0$ or $v_2 \neq 0$. Without loss of generality, assume that $v_1 \neq 0$. This allows us to rewrite the first equation as

$$\lambda = \frac{u_1 - w_1\mu}{v_1}.$$

By plugging this into the second equation and re-grouping, we get

$$\mu\Big(w_2 - \frac{v_2 w_1}{v_1}\Big) + \frac{u_1 v_2}{v_1} = u_2.$$

Now observe that $w_2 - \frac{v_2 w_1}{v_1} = 0$ would imply that $\frac{w_1}{v_1}\mathbf{v} = \mathbf{w}$, which contradicts our assumption. Thus, we conclude that $w_2 - \frac{v_2 w_1}{v_1} \neq 0$ and that we can solve for μ as

$$\mu = \frac{u_2 - \frac{u_1 v_2}{v_1}}{w_2 - \frac{v_2 w_1}{v_1}}.$$

In other words, we proved that choosing $\mu = \frac{u_2 - \frac{u_1 v_2}{v_1}}{w_2 - \frac{v_2 w_1}{v_1}}$ and $\lambda = \frac{u_1 - w_1\mu}{v_1}$ is possible (we had to argue that the denominator is non-zero), and with this choice we get

$$\lambda\mathbf{v} + \mu\mathbf{w} = \mathbf{u},$$

as desired.