D-INFK
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Solution for Assignment 11

1. **a)** Let C_{ij} be the co-factors of A where $i, j \in [5]$. Note that by combining Propositions 6.0.16 and 6.0.9, we get

$$\begin{aligned}\mathsf{det} A &\overset{6.0.9}{=} \mathsf{det} A^\top \\ &\overset{6.0.16}{=} \sum_{j=1}^{5} (A^\top)_{3,j} (C^\top)_{3,j} \\ &= \sum_{i=1}^{5} A_{i,3} C_{i,3} \\ &= 0C_{1,3} + 0C_{2,3} + bC_{3,3} + 0C_{4,3} + 0C_{5,3} \\ &= b \cdot (-1)^{(3+3)} \cdot \begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix}.\end{aligned}$$

This is also sometimes called *expansion of the determinant along the third column.* In particular, we chose the third column because it contains many zeroes and hence many terms disappeared. In order to compute

$$\begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix}$$

we use the same trick again for the first column. In this way we obtain

$$\begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix} = a \cdot (-1)^{(2+1)} \begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix}.$$

We repeat this one more time for the third column of

$$\begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix}$$

to get

$$\begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix} = c \cdot (-1)^{(1+3)} \cdot \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} + 1 \cdot (-1)^{(3+3)} \cdot \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix}.$$

We can compute these 2×2 determinants directly with the formula from the lecture as

$$\begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -2 \text{ and } \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix} = 9.$$

Putting everything together, we obtain

$$\mathsf{det} A = b \cdot (-1)^{(3+3)} \left(a \cdot (-1)^{(2+1)} \left(c \cdot (-1)^{(1+3)} \cdot (-2) + 1 \cdot (-1)^{(3+3)} \cdot 9 \right) \right) = ab(2c - 9).$$

We conclude that $\mathsf{det} A = 0$ if and only if $a = 0$, or $b = 0$, or $c = \frac{9}{2}$.

b) As it turns out, we only need to perform one step of Gauss elimination on B to obtain U:

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & 0 \\ -1 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 6 \\ 0 & 0 & -1 \end{bmatrix} =: U.$$

Using Proposition 6.0.8, we see that $\mathsf{det}(U) = -2$. By using Proposition 6.0.22 (and the discussion in Section 6.0.4), we know that the determinant of U is the same as the determinant of B (we did not swap any rows). Hence, we conclude $\mathsf{det}(B) = -2$.

2. a) This is a solution for a) that uses the decomposition in the hint. Further below, we provide an alternative solution that does not use the hint. Observe that, as suggested in the hint, we can decompose M as

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & C \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

where we used identity matrices I and zero matrices 0 of the appropriate dimensions. By Proposition 6.0.12, we have

$$\mathsf{det}(M) = \mathsf{det}(\begin{bmatrix} I & B \\ 0 & C \end{bmatrix})\mathsf{det}(\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}).$$

Consider first the matrix $M' := \begin{bmatrix} I & B \\ 0 & C \end{bmatrix}$. Observe that there is a unique non-zero entry in each of its first m columns. Thus, every permutation σ that contributes to the determinant of M' in the formula

$$\mathsf{det}M' = \sum_{\sigma \in \Pi_\mathrm{n}} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M'_{i,\sigma(i)}$$

must select these non-zero entries, i.e. $\sigma(i) = i$ for all $i \in [m]$. The formula then simplyfies to

$$\mathsf{det}M' = \sum_{\sigma \in \Pi_\mathrm{n-m}} \mathsf{sign}(\sigma) \prod_{i=1}^{n-m} C_{i,\sigma(i)} = \mathsf{det}C$$

as the non-zero entries in the first m columns (or rows) are all one.

Analogously, we get

$$\mathsf{det} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \sum_{\sigma \in \Pi_\mathrm{m}} \mathsf{sign}(\sigma) \prod_{i=1}^{m} A_{i,\sigma(i)} = \mathsf{det}A$$

and thus we conclude $\mathsf{det}M = \mathsf{det}(A)\mathsf{det}(C)$.

a') Here is a solution that ignores the decomposition in the hint: We start by using the definition of the determinant for M, i.e. we have

$$\mathsf{det}M = \sum_{\sigma \in \Pi_\mathrm{n}} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$$

where Π_n is the set of all permutations on n elements. The key observation for this exercise is that only those permutations $\sigma \in \Pi_\mathrm{n}$ that satisfy $\sigma(1), \ldots, \sigma(m) \in \{1, \ldots, m\}$ will contribute to this sum. To see this, let $\sigma \in \Pi_\mathrm{n}$ be a permutation with $\sigma(i) > m$ for some $i \in [m]$. By the pigeonhole principle, there must exist $j \in [n] \setminus [m]$ with $\sigma(j) \in [m]$. But by the shape of M, we must have $M_{j,\sigma(j)} = 0$ and hence the contribution of σ to the sum is 0.

In particular, the relevant (those that contribute non-zero terms to the sum) permutations $\sigma \in \Pi_\mathrm{n}$ satisfy $\sigma(i) \in [m]$ for all $i \in [m]$ and $\sigma(j) \in [n] \setminus [m]$ for all $j \in [n] \setminus [m]$. In other words, restricting such a permutation σ to $[m]$ yields a permutation on m elements, and restricting σ to $[n] \setminus [m]$ yields a permutation on $n-m$ elements. Conversely, any two permutations $\sigma_1 \in \Pi_{\mathrm{m}}$ and $\sigma_2 \in \Pi_{\mathrm{n-m}}$ yield a permutation $\sigma \in \Pi_{\mathrm{n}}$ that contributes to the sum (define $\sigma(i) = \sigma_1(i)$ for $i \in [m]$ and $\sigma(j) = m + \sigma_2(j-m)$ for $j \in [n] \setminus [m]$). Observe that the number of inversions in σ is exactly the number of inversions in σ_1 plus the number of inversions in σ_2. Hence, we always have $\mathsf{sign}(\sigma) = \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2)$ in this correspondence.

We conclude that we can rewrite the sum as

$$\mathsf{det}M = \sum_{\sigma\in\Pi_{\mathrm{n}}} \mathsf{sign}(\sigma)\prod_{i=1}^{n} M_{i,\sigma(i)} = \sum_{\sigma_1\in\Pi_{\mathrm{m}}}\sum_{\sigma_2\in\Pi_{\mathrm{n-m}}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2)\prod_{i=1}^{m} M_{i,\sigma_1(i)} \prod_{j=m+1}^{n} M_{j,\,j+\sigma_2(j-m)}.$$

Next, observe that the terms $M_{i,\sigma_1(i)}$ are always in the A-part of M, i.e. we have $M_{i,\sigma_1(i)} = A_{i,\sigma_1(i)}$. Similarly, the terms $M_{j,\,j+\sigma_2(j-m)}$ are always in the C-part of M, i.e. we have $M_{j,\,j+\sigma_2(j-m)} = C_{j-m,\sigma_2(j-m)}$. Hence, we can further rewrite the sum as

$$\begin{aligned}
\mathsf{det}M &= \sum_{\sigma_1\in\Pi_{\mathrm{m}}}\sum_{\sigma_2\in\Pi_{\mathrm{n-m}}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2)\prod_{i=1}^{m} M_{i,\sigma_1(i)} \prod_{j=m+1}^{n} M_{j,\,j+\sigma_2(j-m)} \\
&= \sum_{\sigma_1\in\Pi_{\mathrm{m}}}\sum_{\sigma_2\in\Pi_{\mathrm{n-m}}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2)\prod_{i=1}^{m} A_{i,\sigma_1(i)} \prod_{j=m+1}^{n} C_{j-m,\,\sigma_2(j-m)} \\
&= \sum_{\sigma_1\in\Pi_{\mathrm{m}}} \mathsf{sign}(\sigma_1)\prod_{i=1}^{m} A_{i,\sigma_1(i)} \left(\sum_{\sigma_2\in\Pi_{\mathrm{n-m}}} \mathsf{sign}(\sigma_2) \prod_{j=m+1}^{n} C_{j-m,\,\sigma_2(j-m)}\right) \\
&= \left(\sum_{\sigma_1\in\Pi_{\mathrm{m}}} \mathsf{sign}(\sigma_1)\prod_{i=1}^{m} A_{i,\sigma_1(i)}\right) \left(\sum_{\sigma_2\in\Pi_{\mathrm{n-m}}} \mathsf{sign}(\sigma_2) \prod_{j=m+1}^{n} C_{j-m,\,\sigma_2(j-m)}\right) \\
&= \left(\sum_{\sigma_1\in\Pi_{\mathrm{m}}} \mathsf{sign}(\sigma_1)\prod_{i=1}^{m} A_{i,\sigma_1(i)}\right) \left(\sum_{\sigma_2\in\Pi_{\mathrm{n-m}}} \mathsf{sign}(\sigma_2) \prod_{j=1}^{n-m} C_{j,\,\sigma_2(j)}\right) \\
&= \mathsf{det}(A)\mathsf{det}(C)
\end{aligned}$$

which concludes the proof.

b) In order to calculate the determinant of M using the previous result, we must first bring it into the right form. Clearly, M already contains a lot of zero entries. In the end, we want to have a block of zeroes in the bottom left corner. We can use that transposing the matrix does not change its determinant. Moreover, by Proposition 6.0.21, swapping two rows of a matrix negates its determinant. Hence we proceed as follows: we first transpose M and then swap the second row and fourth row, as well as the third and sixth row of the resulting matrix. In this way, we obtain the matrix

$$M' = \begin{bmatrix} 2 & 9 & 1 & 3 & 2 & 8 \\ 4 & 0 & 0 & 5 & 5 & 3 \\ 7 & 4 & 0 & 7 & 2 & 1 \\ 0 & 0 & 0 & 2 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Using the result from the previous subtask and some more row swaps as well as the formula for the determinant of triangular matrices, we get

$$\begin{aligned}
\mathsf{det}M &= (-1)^2\mathsf{det}M' \\
&= \mathsf{det}M' \\
&= \begin{vmatrix} 2 & 9 & 1 \\ 4 & 0 & 0 \\ 7 & 4 & 0 \end{vmatrix} \begin{vmatrix} 2 & 3 & 8 \\ 0 & 0 & 2 \\ 0 & 1 & 7 \end{vmatrix} \\
&= (-1)^2 \begin{vmatrix} 4 & 0 & 0 \\ 7 & 4 & 0 \\ 2 & 9 & 1 \end{vmatrix} (-1) \begin{vmatrix} 2 & 3 & 8 \\ 0 & 1 & 7 \\ 0 & 0 & 2 \end{vmatrix} \\
&= -16 \cdot 4 = -64.
\end{aligned}$$

3. **a)** Using the rules we learned in the lecture, we calculate

$$\begin{aligned}
u + v + w &= (u+v) + w = (4+2i) + (3-4i) = (4+3) + (2-4)i = 7 - 2i \\
u \cdot v &= (3+i)\cdot(1+i) = 3 + 3i + i - 1 = 2 + 4i \\
v \cdot w \cdot i &= (1+i)\cdot(3-4i)\cdot i = (3 - 4i + 3i + 4)\cdot i = 3i + 4 - 3 + 4i = 1 + 7i \\
\frac{w}{v} &= \frac{w}{v}\cdot\frac{\overline{v}}{\overline{v}} = \frac{(3-4i)(1-i)}{(1+i)(1-i)} = \frac{3 - 3i - 4i - 4}{1+1} = -\frac{1}{2} - \frac{7}{2}i \\
\frac{v}{u} &= \frac{v}{u}\cdot\frac{\overline{u}}{\overline{u}} = \frac{(1+i)(3-i)}{(3+i)(3-i)} = \frac{3 - i + 3i + 1}{9+1} = \frac{2}{5} + \frac{1}{5}i \\
|v| &= \sqrt{1^2 + 1^2} = \sqrt{2}.
\end{aligned}$$

4. In this exercise we want to exploit Proposition 6.0.22 which says that the determinant is linear in each row. In particular, using this on the second row of A and B, we get

$$\mathsf{det}(A) - \mathsf{det}(B) = \mathsf{det}\begin{bmatrix} — & \mathbf{v}_1^\top & — \\ — & \mathbf{u}_1^\top & — \\ & M & \end{bmatrix} - \mathsf{det}\begin{bmatrix} — & \mathbf{v}_1^\top & — \\ — & \mathbf{u}_2^\top & — \\ & M & \end{bmatrix} = \mathsf{det}\begin{bmatrix} — & \mathbf{v}_1^\top & — \\ — & (\mathbf{u}_1 - \mathbf{u}_2)^\top & — \\ & M & \end{bmatrix}.$$

Analogously, we can use it on the second row of C and D to get

$$\mathsf{det}(C) - \mathsf{det}(D) = \mathsf{det}\begin{bmatrix} — & \mathbf{v}_2^\top & — \\ — & \mathbf{u}_1^\top & — \\ & M & \end{bmatrix} - \mathsf{det}\begin{bmatrix} — & \mathbf{v}_2^\top & — \\ — & \mathbf{u}_2^\top & — \\ & M & \end{bmatrix} = \mathsf{det}\begin{bmatrix} — & \mathbf{v}_2^\top & — \\ — & (\mathbf{u}_1 - \mathbf{u}_2)^\top & — \\ & M & \end{bmatrix}.$$

Finally, using linearity in the first row of those two resulting matrices yields

$$\mathsf{det}\begin{bmatrix} — & \mathbf{v}_1^\top & — \\ — & (\mathbf{u}_1 - \mathbf{u}_2)^\top & — \\ & M & \end{bmatrix} - \mathsf{det}\begin{bmatrix} — & \mathbf{v}_2^\top & — \\ — & (\mathbf{u}_1 - \mathbf{u}_2)^\top & — \\ & M & \end{bmatrix} = \mathsf{det}\begin{bmatrix} — & (\mathbf{v}_1 - \mathbf{v}_2)^\top & — \\ — & (\mathbf{u}_1 - \mathbf{u}_2)^\top & — \\ & M & \end{bmatrix}$$

and thus

$$\mathsf{det}(A) - \mathsf{det}(B) - \mathsf{det}(C) + \mathsf{det}(D) = \mathsf{det}(E).$$

5. **a)** Let $\lambda \in \mathbb{R}$ be an arbitrary real eigenvalue of M with corresponding real eigenvector $\mathbf{v} \in \mathbb{R}^n$, i.e. we have

$$M\mathbf{v} = \lambda\mathbf{v}.$$

Now let's see what happens to $\mathbf{v}$ if we apply $M + cI$ instead of M to it:

$$\begin{aligned}
(M + cI)\mathbf{v} &= M\mathbf{v} + c\mathbf{v} \\
&= \lambda\mathbf{v} + c\mathbf{v} \\
&= (\lambda + c)\mathbf{v}.
\end{aligned}$$

As we have observed, $\mathbf{v}$ is a real eigenvector of $M + cI$ with corresponding real eigenvalue $c + \lambda$. This is exactly what we wanted to prove.

b) Consider the matrix

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \end{bmatrix}.$$

We observe that $A = B + 2I$. Hence, our plan is to find two distinct real eigenvalues of B, and then use the result from the previous subtask.

Since all rows of B are equal, the matrix has rank 1. Thus, 0 is an eigenvalue of B. It remains to find another real eigenvalue. For this, let us try to guess a real eigenvector of B that does not correspond to eigenvalue 0. This is not as hard as it may sound: every row of B is the same, hence any eigenvector of B that does not correspond to eigenvalue 0 should have the same value in each coordinate. Indeed, we have

$$B \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 36 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the vector $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$ is an eigenvector of B with corresponding eigenvalue 36.

By the result from the previous subtask, it follows that $\lambda_1 = 2$ and $\lambda_2 = 38$ are two distinct real eigenvalues of A.

6. a) Let $\mathbf{x}$ denote the vector of x-coordinates $\mathbf{x} = \begin{bmatrix} p_{x,1} & \dots & p_{x,n} \end{bmatrix}^\top$ and let $\mathbf{y}$ denote the vector of y-coordinates $\mathbf{y} = \begin{bmatrix} p_{y,1} & \dots & p_{y,n} \end{bmatrix}^\top$. The smoothness property can be rewritten as

$$\begin{aligned} &\mathbf{p}_j - \frac{1}{2}(\mathbf{p}_{j-1} + \mathbf{p}_{j+1}) = \mathbf{0} \;\; \forall\, j \in \{2, \dots, n-1\} \\ &\mathbf{p}_1 - \frac{1}{2}(\mathbf{p}_n + \mathbf{p}_2) = \mathbf{0} \\ &\mathbf{p}_n - \frac{1}{2}(\mathbf{p}_{n-1} + \mathbf{p}_1) = \mathbf{0} \end{aligned}$$

which translates to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ with

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

The matrix A can also be written as

$$A = I - \frac{1}{2}(T + E_{1,n}) - \frac{1}{2}(T + E_{1,n})^\top$$

where T is the matrix with ones on the first strict upper diagonal (i.e. the entries where the row coefficient i and column coefficient j satisfy $j = i+1$ for $i \in \{1, ..., n-1\}$) and zeroes everywhere else, and $E_{1,n}$ has a single non-zero entry in row 1 and column n that is equal to 1.

We also want to satisfy the constraints $\mathbf{p}_{j_s} = \mathbf{c}_s$ for all $s \in [k]$. Let $\mathbf{x}^c$ denote the vector of x-coordinates of the locations, i.e. $\mathbf{x}^c = \begin{bmatrix} c_{x,1} & \dots & c_{x,k} \end{bmatrix}^\top$ and let $\mathbf{y}^c$ denote the vector of y-coordinates $\mathbf{y}^c = \begin{bmatrix} c_{y,1} & \dots & c_{y,n} \end{bmatrix}^\top$. Then, the location constraints can be written as $B\mathbf{x} = \mathbf{x}^c$ and $B\mathbf{y} = \mathbf{y}^c$ where the matrix $B \in \mathbb{R}^{k\times n}$ is given by $B_{s,r} = \delta_{r,j_s}$ for all $s \in [k]$ and $r \in [n]$ (recall that the Kronecker-Delta δ_{r,j_s} is one if $r = j_s$ and zero otherwise). In other words, an entry $B_{s,r}$ is one whenever the vertex $\mathbf{p}_r$ should match location $\mathbf{c}_s$ according to the prescribed correspondence $\mathcal{C}$, and $B_{s,r}$ is zero otherwise.

The final systems of linear equations hence are

$$\left[\begin{array}{c} A \\ \hline B \end{array}\right] \mathbf{x} = \left[\begin{array}{c} \mathbf{0_n} \\ \hline \mathbf{x}^c \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} A \\ \hline B \end{array}\right] \mathbf{y} = \left[\begin{array}{c} \mathbf{0_n} \\ \hline \mathbf{y}^c \end{array}\right]$$

where $\mathbf{0}_n$ denotes the n dimensional all-zero vector.

b) Let $S = \left[\begin{array}{c} A \\ \hline B \end{array}\right]$ denote the system matrix. Indeed, the system matrix is the same for both linear systems. Since $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{k\times n}$, the system matrix S is in $\mathbb{R}^{(n+k)\times n}$. This implies that S has rank at most n.

c) We are solving for the curve vertex positions in the least squares sense for the values $n = 6$, $k = 3$, $\mathcal{C} = \{j_1 = 1, j_2 = 3, j_3 = 5\}$ and

$$\mathbf{c}_1 = \begin{bmatrix} c_{x,1} \\ c_{y,1} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_2 = \begin{bmatrix} c_{x,2} \\ c_{y,2} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_3 = \begin{bmatrix} c_{x,3} \\ c_{y,3} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Our strategy is to first combine the two linear systems in one larger system and then solve this using the least squares method. Observe that the two systems

$$\left[\begin{array}{c} A \\ \hline B \end{array}\right] \mathbf{x} = \left[\begin{array}{c} \mathbf{0_n} \\ \hline \mathbf{x}^c \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} A \\ \hline B \end{array}\right] \mathbf{y} = \left[\begin{array}{c} \mathbf{0_n} \\ \hline \mathbf{y}^c \end{array}\right]$$

can be rewritten as

$$M \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} A & 0_{n,n} \\ B & 0_{k,n} \\ 0_{n,n} & A \\ 0_{k,n} & B \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0_n} \\ \mathbf{x}^c \\ \mathbf{0_n} \\ \mathbf{y}^c \end{bmatrix}$$

where M is a $2(n+k) \times 2n$ matrix block matrix (meaning that we put it together from smaller matrices) and $0_{n,n}$ and $0_{k,n}$ are zero-matrices of corresponding dimensions.

The normal equations hence yield

$$M^\top M \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = M^\top \begin{bmatrix} \mathbf{0_n} \\ \mathbf{x}^c \\ \mathbf{0_n} \\ \mathbf{y}^c \end{bmatrix}.$$

Plugging in the values of this specific example for A, B, $\mathbf{x}^c$, and $\mathbf{y}^c$, we get

$$S = \begin{bmatrix} A \\ B \end{bmatrix} = \left[\begin{array}{cccccc} 1 & -1/2 & 0 & 0 & 0 & -1/2 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 1 & -1/2 \\ -1/2 & 0 & 0 & 0 & -1/2 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad \begin{bmatrix} \mathbf{0} \\ \mathbf{x}^c \end{bmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 2 \\ 6 \\ 4 \end{array}\right], \quad \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^c \end{bmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 2 \\ 2 \\ 0 \end{array}\right].$$

The exact final solution is (obtained by solving the normal equations with a computer)

$$\mathbf{x} = \begin{bmatrix} 76/29 & 4 & 156/29 & 148/29 & 4 & 84/29 \end{bmatrix}^\top \text{ and } \mathbf{y} = \begin{bmatrix} 52/29 & 60/29 & 52/29 & 28/29 & 12/29 & 28/29 \end{bmatrix}^\top.$$

A drawing of this solution is provided in Figure 1 below.

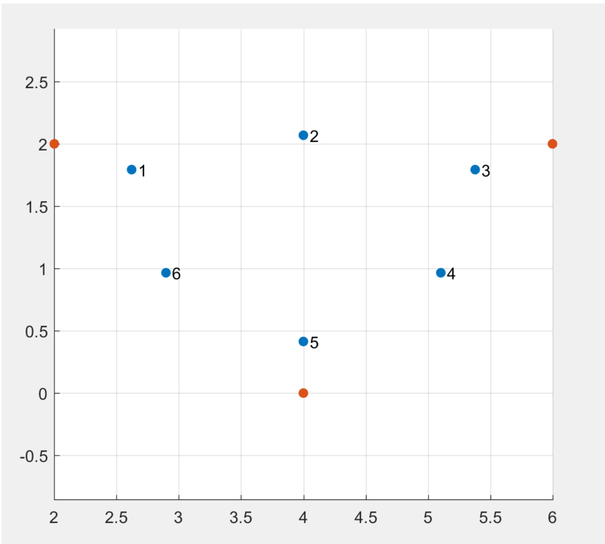

Figure 1: A drawing of the solution.