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Solution for Assignment 6

1. **a)** Since H is a hyperplane, there exists a non-zero vector $\mathbf{d} \in \mathbb{R}^m$ such that $H = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$. In order to prove that H is a subspace of $\mathbb{R}^m$, we have to prove that H is non-empty and closed under vector addition and scalar multiplication. By definition, $\mathbf{0} \in H$ and hence H is non-empty. It remains to prove that, given arbitrary $\mathbf{v}, \mathbf{w} \in H$ and $c \in \mathbb{R}$, we also have $(\mathbf{v} + \mathbf{w}) \in H$ and $c\mathbf{v} \in H$. Indeed, we observe

$$(\mathbf{v} + \mathbf{w}) \cdot \mathbf{d} = \mathbf{v} \cdot \mathbf{d} + \mathbf{w} \cdot \mathbf{d} = 0 + 0 = 0$$

and hence $(\mathbf{v} + \mathbf{w})$ is in H. Similarly, we have

$$(c\mathbf{v}) \cdot \mathbf{d} = c(\mathbf{v} \cdot \mathbf{d}) = c0 = 0$$

and therefore $(c\mathbf{v}) \in H$. We conclude that H is a subspace of $\mathbb{R}^m$.

b) Let $H = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$ for some non-zero $\mathbf{d} \in \mathbb{R}^m$. Consider the standard unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m \in \mathbb{R}^m$ and let $d_i \coloneqq \mathbf{e}_i \cdot \mathbf{d}$ for all $i \in [m]$. Observe that we must have $d_j \neq 0$ for some $j \in [m]$, because $\mathbf{d}$ is non-zero. For every $i \in [m]$, define the vector $\mathbf{v}_i \coloneqq \mathbf{e}_i - \frac{d_i}{d_j}\mathbf{e}_j$. We claim that the set of vectors $\{\mathbf{v}_i : i \neq j\}$ is a basis of H.

To prove this, observe first that the set of vectors $\{\mathbf{v}_i : i \neq j\}$ is linearly independent: Indeed, each of the vectors controls its own coordinate (namely i) and hence there is no way of obtaining the vectors $\mathbf{0}$ by a non-trivial linear combination of vectors in $\{\mathbf{v}_i : i \neq j\}$. Next, observe that, by definition, we have

$$\mathbf{v}_i \cdot \mathbf{d} = \mathbf{e}_i \cdot \mathbf{d} - \frac{d_i}{d_j}\mathbf{e}_j \cdot \mathbf{d} = d_i - d_i = 0$$

for all $i \in [m] \setminus \{j\}$, and thus $\{\mathbf{v}_i : i \neq j\} \subseteq H$. It remains to prove that the vectors span all of H. Let $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \ldots & u_m \end{bmatrix}^\top \in \mathbb{R}^m$ be an arbitrary vector with $\mathbf{u} \cdot \mathbf{d} = \sum_{i=1}^m u_i d_i = 0$. Observe that this implies $\sum_{i \in [m]\setminus\{j\}} u_i d_i = -u_j d_j$. Using this, we get

$$\begin{aligned}
\sum_{i \in [m]\setminus\{j\}} u_i \mathbf{v}_i &= \sum_{i \in [m]\setminus\{j\}} u_i(\mathbf{e}_i - \frac{d_i}{d_j}\mathbf{e}_j) \\
&= \sum_{i \in [m]\setminus\{j\}} u_i \mathbf{e}_i - \big(\sum_{i \in [m]\setminus\{j\}} u_i d_i\big)\frac{1}{d_j}\mathbf{e}_j \\
&= \sum_{i \in [m]\setminus\{j\}} u_i \mathbf{e}_i + u_j \mathbf{e}_j \\
&= \mathbf{u},
\end{aligned}$$

which means that $\mathbf{u} \in \mathsf{Span}\{\mathbf{v}_i : i \neq j\}$. This proves that $\{\mathbf{v}_i : i \neq j\}$ is a basis of H, and we concldue that the dimension of H is $m - 1$.

c) Note that precise notation is very important here: $\mathbf{f} \in V$ is a function and we write $\mathbf{f}(x) \in \mathbb{R}$ for the value of $\mathbf{f}$ evaluated at point $x \in [0, 1]$. In particular, $\mathbf{f}$ and $\mathbf{f}(x)$ are very different types of objects. Note that the symbol $+$ is overloaded in the following sense: for two functions $\mathbf{f}, \mathbf{g} \in V$ and $x \in [0, 1]$, the $+$ in the expression $\mathbf{f}(x) + \mathbf{g}(x)$ denotes the normal addition of real numbers while the $+$ in the expression $\mathbf{f} + \mathbf{g}$ is the addition of functions defined in this exercise.

First, note that U is non-empty since every constant function is in U. Thus, consider arbitrary functions $\mathbf{f}, \mathbf{g} \in U$ and scalar $c \in \mathbb{R}$. For any $x \in [0,1]$, we have

$$(\mathbf{f}+\mathbf{g})(x) \overset{\text{def}}{=} \mathbf{f}(x)+\mathbf{g}(x) \overset{\mathbf{f}\in U}{=} \mathbf{f}(1-x)+\mathbf{g}(x) \overset{\mathbf{g}\in U}{=} \mathbf{f}(1-x)+\mathbf{g}(1-x) \overset{\text{def}}{=} (\mathbf{f}+\mathbf{g})(1-x)$$

and therefore the function $\mathbf{f}+\mathbf{g}$ is in U. Similarly, we have

$$(c\mathbf{f})(x) \overset{\text{def}}{=} c\mathbf{f}(x) \overset{\mathbf{f}\in U}{=} c\mathbf{f}(1-x) \overset{\text{def}}{=} (c\mathbf{f})(1-x)$$

and hence $c\mathbf{f} \in U$. We conclude that U is indeed a subspace of V.

2. Let H be the hyperplane $H = \{\mathbf{u} \in \mathbb{R}^m : \mathbf{u} \cdot \mathbf{v} = 0\}$. By exercise 1b), we know that H has dimension $m-1$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_{m-1} \in H$ be a basis of H. We define the matrices

$$A_i := \begin{bmatrix} \cdots & \mathbf{a}_i^\top & \cdots \\ \cdots & \mathbf{0}^\top & \cdots \end{bmatrix} \in \mathbb{R}^{2\times m}, \quad B_i := \begin{bmatrix} \cdots & \mathbf{0}^\top & \cdots \\ \cdots & \mathbf{a}_i^\top & \cdots \end{bmatrix} \in \mathbb{R}^{2\times m}$$

for every $i \in [m-1]$. Observe that we have $A_i\mathbf{v} = \mathbf{0}$ and $B_i\mathbf{v} = \mathbf{0}$ and thus $A_i, B_i \in S^{\mathbf{v}}$ for all $i \in [m-1]$. We claim that the matrices $A_1, \ldots, A_{m-1}, B_1, \ldots, B_{m-1}$ form a basis of $S^{\mathbf{v}}$. In order to prove this, we first argue that they are linearly independent: For this, consider an arbitrary linear combination

$$\sum_{i=1}^{m-1} \lambda_i A_i + \sum_{i=1}^{m-1} \mu_i B_i = 0$$

with $\lambda_i, \mu_i \in \mathbb{R}$ for all $i \in [m-1]$. Observe that, by definition of $A_1, \ldots, A_{m-1}$ and $B_1, \ldots, B_{m-1}$, this implies $\sum_{i=1}^{m-1} \lambda_i \mathbf{a}_i = \mathbf{0}$ as well as $\sum_{i=1}^{m-1} \mu_i \mathbf{a}_i = \mathbf{0}$. Since $\mathbf{a}_1, \ldots, \mathbf{a}_{m-1}$ are linearly independent, we conclude that $\lambda_1 = \cdots = \lambda_{m-1} = 0$ and $\mu_1 = \cdots = \mu_{m-1} = 0$. Hence, our set of matrices must be linearly independent. It remains to prove that our set of matrices spans $S^{\mathbf{v}}$. For this, let $C \in S^{\mathbf{v}}$ be arbitrary with

$$C = \begin{bmatrix} \cdots & \mathbf{c}_1^\top & \cdots \\ \cdots & \mathbf{c}_2^\top & \cdots \end{bmatrix}.$$

The condition $C\mathbf{v} = \mathbf{0}$ implies $\mathbf{c}_1 \cdot \mathbf{v} = 0$ and $\mathbf{c}_2 \cdot \mathbf{v} = 0$ and therefore $\mathbf{c}_1, \mathbf{c}_2 \in H$. Hence, there exist scalars $\lambda_1, \ldots, \lambda_{m-1} \in \mathbb{R}$ and $\mu_1, \ldots, \mu_{m-1} \in \mathbb{R}$ such that $\sum_{i=1}^{m-1} \lambda_i \mathbf{a}_i = \mathbf{c}_1$ and $\sum_{i=1}^{m-1} \mu_i \mathbf{a}_i = \mathbf{c}_2$. We conclude that

$$C = \sum_{i=1}^{m-1} (\lambda_i A_i + \mu_i B_i)$$

and thus $C \in \mathsf{Span}(A_1, \ldots, A_{m-1}, B_1, \ldots, B_{m-1})$, as desired. This proves that our set of matrices is a basis and we conclude that the dimension of $S^{\mathbf{v}}$ is $2(m-1)$.

3. We want to prove that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

"$\Longleftarrow$" If $U \subseteq W$, then $U \cup W = W$ is a subspace of V by assumption. The same reasoning applies in the case $W \subseteq U$.

"$\Longrightarrow$" Assume now that $U \cup W$ is a subspace of V, and assume that $W \not\subseteq U$ (otherwise, we are done). Then there exists $\mathbf{w} \in W \setminus U$. Let $\mathbf{u} \in U$ be arbitrary. Observe that, since $U \cup W$ is a subspace, we must have $\mathbf{u}+\mathbf{w} \in U \cup W$. But $\mathbf{u}+\mathbf{w} \in U$ would imply that $\mathbf{w} = (\mathbf{u}+\mathbf{w})-\mathbf{u}$ is in U as well. Hence, we conclude $\mathbf{u}+\mathbf{w} \in W$. By $\mathbf{w} \in W$ and $\mathbf{u}+\mathbf{w} \in W$ we finally obtain $\mathbf{u} = (\mathbf{u}+\mathbf{w}) - \mathbf{w} \in W$. We have proven that every vector in U is also in W, and thus conclude $U \subseteq W$.

4. Recall that the dimension of a subspace is defined as the size of a basis of that subspace. So to solve this exercise, it suffices to come up with a basis of $\mathcal{S}_m$. It might be instructive to first consider the case $m = 2$. In the case of $\mathcal{S}_2$, the three matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

form a basis: None of the matrices can be obtained from the others because each of the three matrices has a non-zero entry at a place where none of the other matrices has a non-zero entry (i.e. the three matrices are linearly independent). Moreover, every symmetric 2×2 matrix can be obtained as linear combination of those three matrices because it must have the form

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

for some $a, b, d \in \mathbb{R}$ (i.e. the three matrices span all of $\mathcal{S}_2$).

This idea generalizes to $\mathcal{S}_m$. In particular, for $i, j \in [m]$ with $i \leq j$, define the $m \times m$ matrix $B^{(ij)}$ by

$$B^{(ij)}_{\ell k} = \begin{cases} 1 & \text{if } \ell = i \text{ and } k = j \\ 1 & \text{if } \ell = j \text{ and } k = i \\ 0 & \text{otherwise} \end{cases}$$

for all $\ell, k \in [m]$. Observe that for $i = j$, $B^{(ij)}$ contains a single 1 on its diagonal and is zero everywhere else. For $i < j$, we find exactly two 1s in $B^{(ij)}$ and zeroes everywhere else. We claim that the set of matrices

$$\mathcal{B} = \{B^{(ij)} : i, j \in [m], i \leq j\}$$

is a basis of $\mathcal{S}_m$.

We first check linear independence. Let $i, j \in [m]$ with $i \leq j$ be arbitrary. Then $B^{(ij)}_{ij} = 1$ but none of the other matrices in the set has a non-zero entry at position (i, j). So $B^{(ij)}$ cannot be obtained as a linear combination of the other matrices. We conclude that set of matrices $\mathcal{B}$ is independent.

Let now $S \in \mathcal{S}_m$ be an arbitrary symmetric $m \times m$ matrix. For all $i, j \in [m]$, we must have $S_{ij} = S_{ji}$ by symmetry. Thus, we can write

$$S = \sum_{i,j \in [m] : i \leq j} S_{ij} B^{(ij)}$$

and therefore we conclude that $\mathcal{B}$ spans all of $\mathcal{S}_m$.

Finally, observe that $|\mathcal{B}| = 1 + 2 + 3 + \cdots + m = \frac{m(m+1)}{2}$. Hence, the dimension of $\mathcal{S}_m$ is $\frac{m(m+1)}{2}$.

5. **a)** Note that the function $\mathbf{0} : x \in \mathbb{R} \mapsto 0$ is both in O and E. Hence, both sets are non-empty. Thus, it remains to prove that both O and E are closed under vector addition and scalar multiplication. We start with O. Let $\mathbf{f}, \mathbf{g} \in O$ and $c \in \mathbb{R}$ be arbitrary. We have

$$(\mathbf{f} + \mathbf{g})(-x) = \mathbf{f}(-x) + \mathbf{g}(-x) = -\mathbf{f}(x) - \mathbf{g}(x) = -(\mathbf{g} + \mathbf{f})(x)$$

for all $x \in \mathbb{R}$ and hence $\mathbf{f} + \mathbf{g} \in O$. Similarly, we have

$$(c\mathbf{f})(-x) = c\mathbf{f}(-x) = -c\mathbf{f}(x) = -(c\mathbf{f})(x)$$

for all $x \in \mathbb{R}$ which proves $c\mathbf{f} \in O$. We conclude that O is a subspace of V.

We proceed analogously for E. Let $\mathbf{f}, \mathbf{g} \in E$ and $c \in \mathbb{R}$ be arbitrary. We have

$$(\mathbf{f}+\mathbf{g})(-x) = \mathbf{f}(-x) + \mathbf{g}(-x) = \mathbf{f}(x) + \mathbf{g}(x) = (\mathbf{g}+\mathbf{f})(x)$$

for all $x \in \mathbb{R}$ and hence $\mathbf{f}+\mathbf{g} \in E$. And also

$$(c\mathbf{f})(-x) = c\mathbf{f}(-x) = c\mathbf{f}(x) = (c\mathbf{f})(x)$$

for all $x \in \mathbb{R}$ which proves $c\mathbf{f} \in E$. We conclude that E is a subspace of V.

b) We already know that $\mathbf{0}$ is in both O and E and therefore $\mathbf{0} \in O \cap E$.

Now consider an arbitrary function $\mathbf{f} \in O \cap E$ and fix $x \in \mathbb{R}$. By definition of O, we have $\mathbf{f}(-x) = -\mathbf{f}(x)$. By definition of E, we also get $\mathbf{f}(-x) = \mathbf{f}(x)$. We conclude that we must have $-\mathbf{f}(x) = \mathbf{f}(x)$. But this implies $\mathbf{f}(x) = 0$. Since this works for any $x \in \mathbb{R}$, we conclude that $\mathbf{f}$ must be the zero function $\mathbf{0}$. Hence, $\mathbf{0}$ is the only function in $O \cap E$.

c) Let $\mathbf{f} \in V$ be arbitrary and define

$$\mathbf{g}(x) \coloneqq \frac{1}{2}(\mathbf{f}(x) + \mathbf{f}(-x))$$
$$\mathbf{h}(x) \coloneqq \frac{1}{2}(\mathbf{f}(x) - \mathbf{f}(-x))$$

for all $x \in \mathbb{R}$. Observe that we have $\mathbf{f} = \mathbf{g} + \mathbf{h}$. It remains to prove $\mathbf{g} \in E$ and $\mathbf{h} \in O$: For all $x \in \mathbb{R}$ we have

$$\mathbf{g}(-x) = \frac{1}{2}(\mathbf{f}(-x) + \mathbf{f}(-(-x))) = \frac{1}{2}(\mathbf{f}(x) + \mathbf{f}(-x)) = \mathbf{g}(x)$$

and hence $\mathbf{g} \in E$. Similarly, we have

$$\mathbf{h}(-x) = \frac{1}{2}(\mathbf{f}(-x) - \mathbf{f}(-(-x))) = \frac{1}{2}(\mathbf{f}(-x) - \mathbf{f}(x)) = -\frac{1}{2}(\mathbf{f}(x) - \mathbf{f}(-x)) = -\mathbf{h}(x)$$

for all $x \in \mathbb{R}$. Hence, $\mathbf{h} \in O$.

6. The subspace $\mathsf{Span}(\mathbf{p}, \mathbf{q}, \mathbf{r})$ has dimension 3. We will prove this by showing that $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are linearly independent and hence a basis of $\mathsf{Span}(\mathbf{p}, \mathbf{q}, \mathbf{r})$. For this, let $\lambda, \mu, \gamma \in \mathbb{R}$ such that $\lambda\mathbf{p} + \mu\mathbf{q} + \gamma\mathbf{r} = \mathbf{0}$. If we can prove that this implies $\lambda = \mu = \gamma = 0$, we can conclude that the three polynomials $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are linearly independent.

Since $\mathbf{p}$ is the only one of the tree polynomials involving a non-zero coefficient for x^3, we must have $\lambda = 0$. This further implies that we have $\mu\mathbf{q} + \gamma\mathbf{r} = \mathbf{0}$. Since $\mathbf{r}$ has a non-zero coefficient for x^1, but $\mathbf{p}$ does not, we next observe that we must have $\gamma = 0$. This means that we are left with $\mu\mathbf{q} = \mathbf{0}$ and hence $\mu = 0$.

We conclude that there is no non-trivial linear combination of $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}[x]$ that yields $\mathbf{0} \in \mathbb{R}[x]$. Therefore, the three polynomials are linearly independent. By definition, $\mathbf{p}, \mathbf{q}, \mathbf{r}$ span $\mathsf{Span}(\mathbf{p}, \mathbf{q}, \mathbf{r})$ and we conclude that they are a basis of size 3. This proves that the dimension of this subspace is 3.

7. 1. Let U_1, U_2 be arbitrary subspaces of $\mathbb{R}^m$. Which of the following subsets of $\mathbb{R}^m$ must be subspaces of $\mathbb{R}^m$ as well?

√ **(a)** $U_1 \cap U_2$

Explanation: For vectors $\mathbf{u}, \mathbf{v} \in U_1 \cap U_2$ and a scalar $c \in \mathbb{R}$ we need to prove that $\mathbf{u}+\mathbf{v} \in U_1 \cap U_2$ and $c\mathbf{v} \in U_1 \cap U_2$. By $\mathbf{u}, \mathbf{v} \in U_1 \cap U_2$, we also get $\mathbf{u}, \mathbf{v} \in U_1$ and $\mathbf{u}, \mathbf{v} \in U_2$. Since U_1 and U_2 are subspaces, this implies $\mathbf{u} + \mathbf{v} \in U_1$, $c\mathbf{v} \in U_1$, $\mathbf{u} + \mathbf{v} \in U_2$, and $c\mathbf{v} \in U_2$. Hence, we also have $\mathbf{u} + \mathbf{v} \in U_1 \cap U_2$, and $c\mathbf{v} \in U_1 \cap U_2$.

(b) $U_1 \cup U_2$

Explanation: The set $U_1 \cup U_2$ is in general not a subspace of $\mathbb{R}^m$. For example, U_1 and U_2 could be distinct hyperplanes of $\mathbb{R}^m$. Then, by exercise 1, both U_1 and U_2 are subspaces of $\mathbb{R}^m$ but adding a vector $\mathbf{u}_1 \in U_1$ with a vector $\mathbf{u}_2 \in U_2$ can take us outside of $U_1 \cup U_2$.

(c) $U_1 \setminus U_2 := \{\mathbf{u} \in U_1 : \mathbf{u} \notin U_2\}$

Explanation: The set $U_1 \setminus U_2$ can never be a subspace because the $\mathbf{0}$ is missing.

(d) $\varnothing$

Explanation: By definition, a subspace has to be nonempty.

√ **(e)** $\{\mathbf{0}\}$

Explanation: Adding any two vectors from $\{\mathbf{0}\}$ gives us $\mathbf{0}$ again. Similarly, multiplying with a scalar always gives us $\mathbf{0}$ as well.

√ **(f)** $U_1 + U_2 := \{\mathbf{u}_1 + \mathbf{u}_2 : \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2\}$

Explanation: The set $U_1 + U_2$ is a subspace by design. Consider arbitrary vectors $\mathbf{u}, \mathbf{v} \in U_1 + U_2$ and scalar $c \in \mathbb{R}$. By definition of $U_1 + U_2$, we can write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ with $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$. Similarly, we can write $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1 \in U_1$ and $\mathbf{v}_2 \in U_2$. Then we have

$$\mathbf{u} + \mathbf{v} = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) \in U_1 + U_2$$

and

$$c\mathbf{v} = c(\mathbf{v}_1 + \mathbf{v}_2) = (c\mathbf{v}_1) + (c\mathbf{v}_2) \in U_1 + U_2.$$

2. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

Which of the following sets of vectors is a basis of $\mathbb{R}^4$?

(a)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A set of 5 vectors from $\mathbb{R}^4$ can never be linearly independent. Hence, this is not a basis.

(b)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

The zero vector is linearly dependent on all other vectors. Hence, this is not a basis.

√ **(c)**

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

These 4 vectors are linearly independent. If we put them as columns into a matrix A, then A will have full rank. By the inverse theorem, the system $A\mathbf{x} = \mathbf{b}$ will have a unique solution for all $\mathbf{b} \in \mathbb{R}^4$. Thus, $\mathbf{C}(A) = \mathbb{R}^4$ or in other words, the four vectors span all of $\mathbb{R}^4$. Thus, they are a basis of $\mathbb{R}^4$.

3. Which of the following matrices are in row echelon form?

(a) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Not in row echolon form because of the 2 in the first row.

√ **(b)** $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in row echolon form. There are two pivots, one in the first column and one in the second column.

√ **(c)** $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in row echolon form. The pivots are in the first and third column.

(d) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Not in row echolon form because the 1 and 2 in the first and second row of the third column have not been eliminated.