Bernd Gärtner
Robert Weismantel

Solution for Assignment 7

1. **a)** By using the elimination procedure on A we bring the matrix into reduced row echolon form R:

$$A = \begin{bmatrix} -1 & 2 & 5 & -2 \\ -3 & 3 & 12 & -3 \\ 1 & -14 & -7 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -5 & 2 \\ 0 & -3 & -3 & 3 \\ 0 & -12 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 10 & -20 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} =: R.$$

Performing the same row operations on $\mathbf{b}$ as well yields

$$\mathbf{b} = \begin{bmatrix} -6 \\ -15 \\ 8 \end{bmatrix} \to \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \to \begin{bmatrix} 4 \\ -1 \\ -10 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} =: \mathbf{c}.$$

From the lecture, we know that $A\mathbf{x} = \mathbf{b} \iff R\mathbf{x} = \mathbf{c}$ for all $\mathbf{x} \in \mathbb{R}^4$. The only free variable is x_4. In particular, we can rewrite our system as

$$I_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{c} - \begin{bmatrix} -6 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} x_4 \end{bmatrix} = \mathbf{c} - x_4 \begin{bmatrix} -6 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 + 6x_4 \\ -x_4 \\ -1 + 2x_4 \end{bmatrix}$$

where $\begin{bmatrix} -6 \\ 1 \\ -2 \end{bmatrix} =: F$ (so that we can compare with the explanation in the blackboard notes).

Therefore, the full set of solutions is $\mathcal{L} = \{ \begin{bmatrix} 1 + 6x_4 \\ -x_4 \\ -1 + 2x_4 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_4 \in \mathbb{R}\}$.

b) The nullspace of A contains the solutions to $A\mathbf{x} = \mathbf{0}$. Equivalently, these are the solutions to $R\mathbf{x} = \mathbf{0}$ with the R from the previous subtask. As above, we can rearrange this system to

$$I_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} - \begin{bmatrix} -6 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} x_4 \end{bmatrix} = -x_4 \begin{bmatrix} -6 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6x_4 \\ -x_4 \\ 2x_4 \end{bmatrix}$$

where we again have $\begin{bmatrix} -6 & 1 & -2 \end{bmatrix}^\top = F$. Following the blackboard notes, we can obtain one basis vector of $\mathbf{N}(A)$ from each free variable. In this case, we only have a single free variable. Setting it to 1 yields the solution $\mathbf{x} = \begin{bmatrix} 6 & -1 & 2 & 1 \end{bmatrix}^\top$. We conclude that this single vector is a basis of $\mathbf{N}(A)$. In words, this means that every vector in $\mathbf{N}(A)$ can be obtained from this basis vector.

Now consider the column space of A. By definition, it is spanned by the columns of A. In order to find a basis for it, we have to find an independent subset of these columns that still spans the same space. Equivalently, we have to find the CR decomposition of A. Luckily, we already found R. Moreover, we know from the lecture (Section 3.2.2 in the blackboard notes) that C can now be found by taking those columns in A that have a pivot in R. Concretely, in our case R has pivots in the first three columns. Hence, we get

$$C = \begin{bmatrix} -1 & 2 & 5 \\ -3 & 3 & 12 \\ 1 & -14 & -7 \end{bmatrix}.$$

The columns $\mathbf{v}_1 = \begin{bmatrix} -1 & -3 & 1 \end{bmatrix}^\top, \mathbf{v}_2 = \begin{bmatrix} 2 & 3 & -14 \end{bmatrix}^\top, \mathbf{v}_3 = \begin{bmatrix} 5 & 12 & -7 \end{bmatrix}^\top$ of C are a basis of $\mathbf{C}(A)$.

c) Recall that the dimension of a subspace is the size of its basis. For $\mathbf{N}(A)$, we got 1 basis vector and hence $\mathbf{N}(A)$ has dimension 1. Similarly, $\mathbf{C}(A)$ has dimension 3. In particular, A has rank $r = 3$ and this allows us to calculate the dimensions of $\mathbf{C}(A^\top)$ and $\mathbf{N}(A^\top)$ using the formulas from the lecture as well: the dimension of $\mathbf{C}(A^\top)$ is $r = 3$ while the dimension of $\mathbf{N}(A^\top)$ is $m - r = 0$ where m is the number of rows of A.

d) Recall that row operations preserve the row space and hence we have $\mathbf{R}(A) = \mathbf{R}(R)$. Moreover, all rows of R are linearly independent by construction. Hence, the rows of R form a basis of $\mathbf{R}(A)$. Concretely, a basis of $\mathbf{R}(A)$ is given by the three vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \in \mathbb{R}^4.$$

2. **a)** The key insight here is to use $\mathbf{v}^\top \mathbf{v} = \mathbf{v} \cdot \mathbf{v} = \| \mathbf{v} \|^2 = 1$. We then get

$$A^2 = (\mathbf{v}\mathbf{v}^\top)(\mathbf{v}\mathbf{v}^\top) = \mathbf{v}(\mathbf{v}^\top\mathbf{v})\mathbf{v}^\top = \mathbf{v}1\mathbf{v}^\top = \mathbf{v}\mathbf{v}^\top = A$$

for A^2 and therefore

$$P^2 = (I - A)^2 = I^2 - 2A + A^2 = I - 2A + A = I - A = P$$

for P^2.

b) Knowing that $\mathbf{w} \cdot \mathbf{v} = 0$, we compute $A\mathbf{w} = (\mathbf{v}\mathbf{v}^\top)\mathbf{w} = \mathbf{v}(\mathbf{v}^\top\mathbf{w}) = \mathbf{v}(\mathbf{v} \cdot \mathbf{w}) = (\mathbf{v} \cdot \mathbf{w})\mathbf{v} = 0\mathbf{v} = \mathbf{0}$.

c) We are given $A\mathbf{w} = \mathbf{0}$ and hence get $\mathbf{0} = A\mathbf{w} = (\mathbf{v}\mathbf{v}^\top)\mathbf{w} = \mathbf{v}(\mathbf{v}^\top\mathbf{w}) = \mathbf{v}(\mathbf{v} \cdot \mathbf{w}) = (\mathbf{v} \cdot \mathbf{w})\mathbf{v}$. Note that $\mathbf{v} \cdot \mathbf{w}$ is a scalar and that $\mathbf{v} \neq \mathbf{0}$. From $(\mathbf{v} \cdot \mathbf{w})\mathbf{v} = \mathbf{0}$ we hence get $\mathbf{v} \cdot \mathbf{w} = 0$, as desired.

d) Combining subtasks b) and c), we get that for all $\mathbf{w} \in \mathbb{R}^3$, we have $A\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. Hence, the nullspace of A is given by $\mathbf{N}(A) = \{\mathbf{w} \in \mathbb{R}^3 : \mathbf{w} \cdot \mathbf{v} = 0\}$. In words, this is the set of all vectors orthogonal to $\mathbf{v}$, which is a hyperplane.

e) Observe that A has the form

$$A = \begin{bmatrix} | & | & | \\ v_1\mathbf{v} & v_2\mathbf{v} & v_3\mathbf{v} \\ | & | & | \end{bmatrix}$$

where $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top$. Hence, we have $\mathbf{C}(A) = \mathsf{Span}(\mathbf{v})$. By $\mathbf{v} \neq \mathbf{0}$, we conclude that A has rank 1. Therefore, it is not invertible. Note that we have studied matrices of rank 1 before in Exercise 3 of Assignment 2.

f) We first prove $\mathbf{C}(A) \subseteq \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$. Let $\mathbf{w} \in \mathbf{C}(A)$ be arbitrary. By definition of $\mathbf{C}(A)$, there exists $\mathbf{x}$ with $A\mathbf{x} = \mathbf{w}$. We conclude $\mathbf{w} = A\mathbf{x} = \mathbf{v}\mathbf{v}^\top\mathbf{x} = (\mathbf{v}\cdot\mathbf{x})\mathbf{v} = \alpha\mathbf{v}$ for $\alpha = \mathbf{v}\cdot\mathbf{x}$. Thus, $\mathbf{w} \in \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$.

It remains to prove $\{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} \subseteq \mathbf{C}(A)$. Let $\mathbf{w} \in \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$ be arbitrary, i.e. $\mathbf{w} = \alpha\mathbf{v}$ for some $\alpha \in \mathbb{R}$. Choosing $\mathbf{x} = \mathbf{w}$, we get

$$A\mathbf{x} = \mathbf{v}\mathbf{v}^\top\mathbf{x} = \mathbf{v}\mathbf{v}^\top(\alpha\mathbf{v}) = \alpha(\mathbf{v}\cdot\mathbf{v})\mathbf{v} = \alpha\mathbf{v} = \mathbf{w}$$

and therefore $\mathbf{w} \in \mathbf{C}(A)$.

g) By subtask f), it suffices to prove $\{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$. To see that $\{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} \subseteq \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$, observe that $A(\alpha\mathbf{v}) = \alpha\mathbf{v}$ (we have calculated this in more detail already above) and hence $\alpha\mathbf{v} \in \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$ for all α. It remains to prove $\{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\} \subseteq \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$. By definition, any $\mathbf{w} \in \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$ is also in $\mathbf{C}(A)$. Hence, $\{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\} \subseteq \mathbf{C}(A)$ and with subtask f) we conclude the proof.

h) Let $\mathbf{w} \in \mathbb{R}^3$ be arbitrary. We have

$$P\mathbf{w} = \mathbf{0} \iff (I - A)\mathbf{w} = \mathbf{0} \iff \mathbf{w} - A\mathbf{w} = \mathbf{0} \iff A\mathbf{w} = \mathbf{w}.$$

This implies $\mathbf{N}(P) = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$ and by subtask g) we conclude $\mathbf{N}(P) = \mathbf{C}(A)$.

i) We start by proving $\mathbf{C}(P) \subseteq \mathbf{N}(A)$. Let $\mathbf{w} \in \mathbf{C}(P)$ be arbitrary. By definition, there is $\mathbf{x} \in \mathbb{R}^3$ with $P\mathbf{x} = \mathbf{w}$. We use this to compute $A\mathbf{w}$ as

$$A\mathbf{w} = A(P\mathbf{x}) = A(I - A)\mathbf{x} = (A - A^2)\mathbf{x} = (A - A)\mathbf{x} = \mathbf{0}$$

and conclude $\mathbf{w} \in \mathbf{N}(A)$.

It remains to prove $\mathbf{N}(A) \subseteq \mathbf{C}(P)$. Let $\mathbf{w} \in \mathbf{N}(A)$ be arbitrary. By definition, we have $A\mathbf{w} = \mathbf{0}$. Choosing $\mathbf{x} = \mathbf{w}$, we get $P\mathbf{x} = (I - A)\mathbf{x} = \mathbf{x} - A\mathbf{x} = \mathbf{w} - A\mathbf{w} = \mathbf{w}$. We conclude $\mathbf{w} \in \mathbf{C}(P)$.

3. **a)** Let $\mathbf{v}_1, \mathbf{v}_2$ denote the columns of A. We can rewrite

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix} \text{ to } 1\mathbf{v}_1 + 0\mathbf{v}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

which immediately yields $\mathbf{v}_1 = \begin{bmatrix}1 & 1 & 2\end{bmatrix}^\top$. Similarly, we get

$$A\begin{bmatrix}1\\1\end{bmatrix} = 1\mathbf{v}_1 + 1\mathbf{v}_2 = \begin{bmatrix}2\\3\\2\end{bmatrix}$$

and hence

$$\mathbf{v}_2 = \begin{bmatrix}2\\3\\2\end{bmatrix} - \mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}.$$

We conclude that

$$A = \begin{bmatrix}1 & 1\\1 & 2\\2 & 0\end{bmatrix}.$$

b) Observe that A has $m = 3$ rows, $n = 2$ columns, and rank $r = 2$ (since the two columns of A are linearly independent). From the lecture, we know that the dimension of $\mathbf{C}(A)$ is $r = 2$, the dimension of $\mathbf{C}(A^\top)$ is $r = 2$, the dimension of $\mathbf{N}(A)$ is $n - r = 0$, and the dimension of $\mathbf{N}(A^\top)$ is $m - r = 1$.

4. For every matrix $A = [a_{ij}]_{i=1,j=1}^{m,\ m} \in \mathbb{R}^{m\times m}$, let $\mathsf{flatten}(A) \in \mathbb{R}^{m^2}$ denote the vector

$$\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \\ a_{21} \\ a_{22} \\ \vdots \\ a_{mm} \end{bmatrix} \in \mathbb{R}^{m^2}$$

obtained by concatenating the row vectors of A. Now consider the vector $\mathsf{flatten}(I) \in \mathbb{R}^{m^2}$ obtained by flattening the identity matrix. Observe that, for all $A \in \mathbb{R}^{m\times m}$, we have $\mathsf{Tr}(A) = 0$ if and only if $\mathsf{flatten}(A) \cdot \mathsf{flatten}(I) = 0$, where $\cdot$ denotes the scalar product in $\mathbb{R}^{m^2}$. Thus, we can rewrite S as

$$S = \{A \in \mathbb{R}^{m\times m} \ : \ \mathsf{flatten}(A) \cdot \mathsf{flatten}(I) = 0\}.$$

Looking at this definition of S, we can see that it is a hyperplane of $\mathbb{R}^{m^2}$ (since $\mathsf{flatten}(I) \neq \mathbf{0}$). Using our insights from assignment 6, we thus conclude that the dimension of S is $m^2 - 1$.

5. a) Assume that S is not a single point and not a triangle. We will prove that it then has to be a line segment. Since S is not a single point, two of its vertices must be distinct. Without loss of generality, assume it is $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e. $\mathbf{v}_1 \neq \mathbf{v}_2$. We also know that S is not a triangle. Hence, either we immediately get that S is a line segment and we are done, or we have that not all three vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are distinct. Without loss of generality assume $\mathbf{v}_3 = \mathbf{v}_2$. Then S can be described by just using $\mathbf{v}_1$ and $\mathbf{v}_2$ as

$$\begin{aligned} S &= \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_0^+, \lambda_1 + \lambda_2 + \lambda_3 = 1\} \\ &= \{\lambda_1\mathbf{v}_1 + (\lambda_2 + \lambda_3)\mathbf{v}_2 : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_0^+, \lambda_1 + (\lambda_2 + \lambda_3) = 1\} \\ &= \{\lambda_1\mathbf{v}_1 + \lambda_{23}\mathbf{v}_2 : \lambda_1, \lambda_{23} \in \mathbb{R}_0^+, \lambda_1 + \lambda_{23} = 1\}. \end{aligned}$$

Notice that we still have $\mathbf{v}_1 \neq \mathbf{v}_2$ and hence S is a line segment.

b) Observe that A has rank 0 if and only if it is 0, i.e. $A = 0$. But by definition, this is equivalent to saying that $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3$ which is again equivalent to S being a single point. This proves both directions.

c) We prove both directions individually.

"$\Longrightarrow$" Assume that S is not a triangle. Similar to the argument form a), this implies that we can express one of the three vectors as a convex combination of the others. Assume first that it is $\mathbf{v}_1$, i.e. we have

$$\mathbf{v}_1 = \mu_1\mathbf{v}_2 + \mu_2\mathbf{v}_3$$

with $\mu_1, \mu_2 \in \mathbb{R}_0^+$ such that $\mu_1 + \mu_2 = 1$. Then $\mu_1(\mathbf{v}_2 - \mathbf{v}_1) + \mu_2(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$ is a non-trivial linear combination of the columns of A proving that it cannot have rank 2. Thus, assume now instead that $\mathbf{v}_2$ can be written as convex combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, i.e.

$$\mathbf{v}_2 = \mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_3$$

with $\mu_1, \mu_2 \in \mathbb{R}_0^+$ such that $\mu_1 + \mu_2 = 1$. In this case,

$$-(\mathbf{v}_2 - \mathbf{v}_1) + \mu_2(\mathbf{v}_3 - \mathbf{v}_1) = -\mathbf{v}_2 + \mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_3 = \mathbf{0}$$

is a non-trivial linear combination of the columns of A. The case where $\mathbf{v}_3$ can be written as convex combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is symmetric. In all three cases, we get that A cannot have rank 2. This concludes the argument.

"$\Longleftarrow$" Assume now that A does not have rank 2. Without loss of generality, assume that there exists $\lambda \in \mathbb{R}$ with $\lambda(\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{v}_3 - \mathbf{v}_1$. We distinguish cases based on the value of λ:

- If $\lambda \geq 1$, we can write $\mathbf{v}_2$ as convex combination $\mathbf{v}_2 = \frac{1}{\lambda}\mathbf{v}_3 + (1 - \frac{1}{\lambda})\mathbf{v}_1$. Thus S is not a triangle.
- If $1 > \lambda > 0$, we can write $\mathbf{v}_3$ as convex combination $\mathbf{v}_3 = \lambda\mathbf{v}_2 + (1-\lambda)\mathbf{v}_1$. Thus S is not a triangle.
- If $\lambda = 0$, we have $\mathbf{v}_3 = \mathbf{v}_1$ and immediately conclude that S is not a triangle.
- If $\lambda < 0$, we can write $\mathbf{v}_1$ as convex combination $\mathbf{v}_1 = -\frac{\lambda}{1-\lambda}\mathbf{v}_2 + \frac{1}{1-\lambda}\mathbf{v}_3$. Thus S is not a triangle.

This concludes the argument.

d) We prove both directions individually.

"$\Longrightarrow$" Assume that A has rank 1. By subtasks b) and c) we get that S cannot be a single point and it also cannot be a triangle. Hence, S must be a line segment by subtask a).

"$\Longleftarrow$" Assume that S is a line segment. By subtasks b) and c) we know that A cannot have rank 0 and it also cannot have rank 2. We conclude that is must have rank 1.

6. 1. Which of the following statements is true for all $m \times m$ matrices A?

√ **(a)** $\mathbf{N}(A) = \mathbf{N}(2A)$

Explanation: We know from the lecture that the nullspace of an $m \times m$ matrix is a subspace of $\mathbb{R}^m$. In particular, any nullspace is closed under scalar multiplication. Therefore, $\mathbf{N}(A) = \mathbf{N}(2A)$.

(b) $\mathbf{N}(A) = \mathbf{N}(A^2)$

Explanation: Let $\mathbf{x} \in \mathbb{R}^m$ be arbitrary. If we have $A\mathbf{x} = \mathbf{0}$, we also get $A^2\mathbf{x} = \mathbf{0}$. But the converse is not necessarily true. Consider the 2×2 matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Clearly, there exists $\mathbf{x} \in \mathbb{R}^2$ with $A\mathbf{x} \neq \mathbf{0}$ and hence $\mathbf{N}(A) \neq \mathbb{R}^2$. But we have $A^2 = 0$ and therefore $\mathbf{N}(A^2) = \mathbb{R}^2$.

(c) $\mathbf{N}(A) = \mathbf{N}(A + I)$

Explanation: Consider again the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The matrix $A + I$ has rank 2 while the matrix A has rank 1. Therefore, their nullspaces cannot be the same.

(d) $\mathbf{N}(A) = \mathbf{N}(A^\top)$

Explanation: For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we get $A^\top = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Now consider the standard unit vector $\mathbf{e}_2 \in \mathbb{R}^2$. We have $A\mathbf{e}_2 = \mathbf{e}_1 \neq \mathbf{0}$ and $A^\top\mathbf{e}_2 = \mathbf{0}$ and therefore $\mathbf{N}(A) \neq \mathbf{N}(A^\top)$.

2. Which of the following statements is true for all square matrices A?

√ **(a)** $\mathbf{C}(A) = \mathbf{C}(2A)$

Explanation: We know that the column space of an $m \times m$ matrix is a subspace of $\mathbb{R}^m$. In particular, any column space is closed under scalar multiplication. Therefore, $\mathbf{C}(A) = \mathbf{C}(2A)$.

(b) $\mathbf{C}(A) = \mathbf{C}(A^2)$

Explanation: For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we get $A^2 = 0$. As before, this implies that A has rank 1 while A^2 has rank 0 and hence they have different column spaces (the dimensions of the column spaces must be different, so the spaces themselves must be different as well).

(c) $\mathbf{C}(A) = \mathbf{C}(A + I)$

Explanation: For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we get that A has rank 1 while $A + I$ has rank 2. Therefore, the two column spaces must be different.

(d) $\mathbf{C}(A) = \mathbf{C}(A^\top)$

Explanation: The column space of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by the standard unit vector $\mathbf{e}_1$ while the column space of $A^\top$ is spanned by the standard unit vector $\mathbf{e}_2$. In particular, we have $\mathbf{e}_1 \in \mathbf{C}(A)$ but $\mathbf{e}_1 \notin \mathbf{C}(A^\top)$.

3. The following equations each describe a plane in $\mathbb{R}^3$:

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 0 \\ 2x & - & 5y & + & 3z & = & 0 \\ 3x & & & + & 4z & = & 0. \end{array}$$

Which of the following statements is true?

(a) The intersection of all three planes is empty.

√ **(b)** The intersection of all three planes contains exactly one element.

(c) The intersection of all three planes is a line.

Explanation: For a point to be in the intersection of all three planes, it has to be a solution to all three equations. Thus, we want to understand the set of solutions of the linear system

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -5 & 3 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}.$$

As it turns out, A has full rank (i.e. its rank is 3) and hence this linear system has a unique solution (which is the zero-vector). Therefore, the intersection of all three planes contains exactly one element.

4. Consider the linear system

$$x_1 + (b-1)x_2 = 3$$
$$-3x_1 - (2b-8)x_2 = -5$$

with variables x_1, x_2 and parameter $b \in \mathbb{R}$. For which values of b is the set of solutions to the above system empty (i.e. there is no solution)?

(**a**) Only for $b = 0$.

√ (**b**) Only for $b = -5$.

(**c**) For all possible values of b (i.e. for all of $\mathbb{R}$).

(**d**) The system always has a solution regardless of the value of b.

Explanation: Adding the first equation 3 times to the second equation, we get $(3b-3-2b+8)x_2 = 4$ and thus $(b+5)x_2 = 4$. This equation has a solution whenver $b \neq -5$. But there is no solution if $b = -5$.