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Lecture 21: Projections of sets and the Farkas Lemma

The strategy

The guiding question

Suppose we are given a set of linear inequalities in $\mathbb{R}^n$. How can we certify that the set is nonempty?

Definition (Projection of a set of inequalities)

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ and $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. P is called a polyhedron. Let $S = \{1, \ldots, s\}$. The **projection of P** on the subspace $\mathbb{R}^s$ associated with the variables in the subset S is

$$\text{proj}_S(P) := \{x \in \mathbb{R}^s \mid \exists y \in \mathbb{R}^{n-s} \text{ such that } (x^T, y^T)^T \in P\}.$$

Remark / Question

- $P \neq \emptyset$ if and only if $\text{proj}_S(P) \neq \emptyset$.
- Does $\text{proj}_S(P)$ have a description in form of a finite system of linear inequalities?
- If so, then the question whether $P \neq \emptyset$ is reduced to a question of the same form in smaller dimension.

The one-dimensional case

Intuition

Let $a \in \mathbb{Q}^m$, $a_i \neq 0$ for all i and $b \in \mathbb{Q}^m$. We consider $P = \{x \in \mathbb{R} \mid ax \leq b\} \subseteq \mathbb{R}$. We first notice that we can rewrite the constrains in P as follows. Set

$$u := \min\{\frac{b_i}{a_i} \mid a_i > 0\}, \quad l := \max\{\frac{b_i}{a_i} \mid a_i < 0\}.$$

$$P = \{x \in \mathbb{R} \mid x \leq \frac{b_i}{a_i} \text{ if } a_i > 0,\ x \geq \frac{b_i}{a_i} \text{ if } a_i < 0\} = \{x \in \mathbb{R} \mid x \leq u,\ x \geq l\}.$$

Proposition

$P \neq \emptyset \iff l \leq u \iff 0 \leq u - l \iff 0 \leq y^T b$ for all $y \geq 0$ such that $y^T a = 0$.

We want to derive such a result in general dimensions!

Let $A \in \mathbb{Q}^{m \times n}$ with entries a_{ij}, let $b \in \mathbb{Q}^m$ and $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$.
Let $\bar{x} = (x_1, \ldots, x_{n-1})$ and $\bar{A} = [A_{\cdot 1} \ldots A_{\cdot n-1}]$.

Elimination of one variable

Algorithm

(1) Partition the indices $M = \{1, \ldots, m\}$ of the rows of A into three subsets

$$M_0 = \{i \in M \mid a_{i,n} = 0\}, \quad M_+ = \{i \in M \mid a_{i,n} > 0\} \text{ and } M_- = \{i \in M \mid a_{i,n} < 0\}.$$

(2)
- For every row with index $i \in M_+$ multiply the corresponding constraint by $\frac{1}{a_{in}}$.

$$x_n \leq d_i + f_i^T \bar{x} \text{ for } i \in M_+ \text{ where } d_i = \frac{b_i}{a_{in}},\ f_{ij} = -\frac{a_{ij}}{a_{in}}.$$

- Every row with index $k \in M_0$ can be rewritten as

$$0 \leq d_k + f_k^T \bar{x} \text{ for } k \in M_0 \text{ where } d_k = b_k,\ f_{kj} = -a_{kj}.$$

- For every row with index $i \in M_-$ multiply the corresponding constraint by $\frac{1}{a_{in}}$.

$$x_n \geq d_i + f_i^T \bar{x} \text{ for } i \in M_- \text{ where } d_i = \frac{b_i}{a_{in}},\ f_{ij} = -\frac{a_{ij}}{a_{in}}.$$

(3) Return Q.

Elimination of one variable continued

$$Q = \Big\{ \bar{x} \in \mathbb{R}^{n-1} \mid \quad 0 \leq \quad d_k + f_k^T \bar{x} \text{ for all } k \in M_0, \\ d_l + f_l^T \bar{x} \quad \leq \quad d_i + f_i^T \bar{x} \text{ for all } l \in M_-,\ i \in M_+ \Big\}.$$

Theorem 3. Let $S = \{1, \ldots, n-1\}$

The set Q returned in Step 3 is a polyhedron. Moreover, $Q = \text{proj}_S(P)$.

Proof of the first statement

- Q is a polyhedron, because we find $F \in \mathbb{Q}^{k \times n-1}$ and $f \in \mathbb{Q}^k$ such that

$$Q = \Big\{ \bar{x} \in \mathbb{R}^{n-1} \mid F\bar{x} \leq f \Big\}.$$

- Let $k = |M_0| + |M_-||M_+|$. The rows of F contain all rows of A with index $i \in M_0$. The corresponding right hand side vector satisfies that $f_i = b_i$.
- The other rows of F are of the form $(f_l - f_i)^T$ for indices $l \in M_-$ and $i \in M_+$. The corresponding right hand side entry of f is then $d_i - d_l$.

Proof continued

$\text{proj}_S(P) \subseteq Q$.

Take any $\bar{x} \in \text{proj}_S(P)$. There exists $z \in \mathbb{R}$ such that $(\bar{x}, z) \in P$. Hence z satisfies the constraints in Step 2. In particular,

$$d_l + f_l^T \bar{x} \leq z \leq d_i + f_i^T \bar{x} \text{ for all } l \in M_-,\ i \in M_+.$$

This shows that $\bar{x} \in Q$.

$Q \subseteq \text{proj}_S(P)$

Take any $\bar{x} \in Q$. It follows that

$$\left.\begin{array}{rcl} 0 & \leq & d_k + f_k^T \bar{x} \text{ for all } k \in M_0, \\ d_l + f_l^T \bar{x} & \leq & d_i + f_i^T \bar{x} \text{ for all } l \in M_-,\ i \in M_+ \end{array}\right\}.$$

Let $L := \max\{d_l + f_l^T \bar{x} \mid l \in M_-\}$ and $U := \min\{d_i + f_i^T \bar{x} \mid i \in M_+\}$. Take any value $z \in [L, U]$. Then $(\bar{x}, z) \in P$. Hence, $\bar{x} \in \text{proj}_S(P)$.

Use these projections repeatedly

Lemma

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$ *and* $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$. *For indices* $1 \le k < j < n$, *consider*

$$S_1 = \{1,\dots,n-k\} \text{ and } S_2 = \{1,\dots,n-j\}.$$

Then $\text{proj}_{S_2}(P) = \text{proj}_{S_2}(\text{proj}_{S_1}(P))$.

Proof for $k = 1$ and $j = 2$

- Let $z \in \text{proj}_{S_2}(P)$. There exist $(x_{n-1}, x_n) \in \mathbb{R}^2$ such that $(z, x_{n-1}, x_n) \in P$. In particular, there exists a value x_n such that

$$(z, x_{n-1}, x_n) \in P \quad \Rightarrow \quad (z, x_{n-1}) \in \text{proj}_{S_1}(P) \quad \Rightarrow \quad z \in \text{proj}_{S_2}(\text{proj}_{S_1}(P)).$$

- Conversely, take $z \in \text{proj}_{S_2}(\text{proj}_{S_1}(P))$, i.e., there exists $x_{n-1} \in \mathbb{R}$ such that

$$(z, x_{n-1}) \in \text{proj}_{S_1}(P).$$

  Hence there exists $x_n \in \mathbb{R}$ such that $(z, x_{n-1}, x_n) \in P$, i.e., $z \in \text{proj}_{S_2}(P)$.

The elimination process algebraically

Definition 5

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ and $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$.
We define $A^{(j)}$ to be the submatrix of A with column vectors $A_{\cdot,k}$ for $k = 1, \ldots, j$. Let $P^{(0)} = P$ and $C^{(0)} = \mathbb{R}^m_+$.

Define for $i \in \{1, \ldots, n\}$

$$C^{(i)} = \left\{ y \in \mathbb{R}^m_+ \mid y^T A_{\cdot k} = 0 \text{ for all } k = n-i+1, \ldots, n \right\}.$$

$$P^{(i)} = \left\{ \bar{x} \in \mathbb{R}^{n-i} \mid y^T A^{(n-i)} \bar{x} \leq y^T b \text{ for all } y \in C^{(i)} \right\}.$$

Theorem

- $\text{proj}_{S_{n-i}}(P) = P^{(i)}$.
- *The proof shows that $P^{(i)}$ is a polyhedron*
- *Polyhedra are closed under projections.*

First part of the proof

$\text{proj}_{S_{n-i}}(P) \subseteq P^{(i)}$

- Let $\bar{x} \in \text{proj}_{S_{n-i}}(P)$. By definition, there exists $z \in \mathbb{R}^i$ such that

$$(\bar{x}, z) \in P.$$

- Hence, $(\bar{x}, z)$ satisfies the following inequalities

$$\sum_{k=1}^{n-i} A_{.k}\bar{x}_k + \sum_{k=n-i+1}^{n} A_{.k}z_k \leq b.$$

- This implies that for all $y \in C^{(i)}$ we obtain that

$$\begin{aligned} \textstyle\sum_{k=1}^{n-i} y^T A_{.k}\bar{x}_k + \sum_{k=n-i+1}^{n} y^T A_{.k}z_k &= \textstyle\sum_{k=1}^{n-i} y^T A_{.k}\bar{x}_k \\ &= y^T A^{(n-i)}\bar{x} \leq y^T b, \end{aligned}$$

- i.e., $\bar{x} \in P^{(i)}$.

Second part of the proof

$P^{(i)} \subseteq \text{proj}_{S_{n-i}}(P)$.

- We apply an inductive argument. The base case is $i = 1$. Recall

$$\begin{aligned} C^{(1)} &= \left\{ y \in \mathbb{R}^m_+ \mid y^T A_{\cdot n} = 0 \right\}, \\ P^{(1)} &= \left\{ \bar{x} \in \mathbb{R}^{n-1} \mid y^T A^{(n-1)} \bar{x} \leq y^T b \text{ for all } y \in C^{(1)} \right\}. \end{aligned}$$

- $P^{(1)} \subseteq \text{proj}_{S_{n-1}}(P)$ follows by observing:
  - if one takes y as the unit vector e_k for $k \in M_0$. Then $e_k \in C^{(1)}$.
  - pick two indices $l \in M_-$ and $i \in M_+$. Then

$$y := -\frac{1}{a_{ln}} e_l + \frac{1}{a_{in}} e_i \in C^{(1)}.$$

- The corresponding inequalities obtained from choosing y as described above are part of the description of $Q = \text{proj}_{S_{n-1}}(P)$ in Theorem 3.
- The inductive step can be shown similarly.

Farkas Lemma

Theorem (The Farkas Lemma)

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$. Either there exists a vector $x \in \mathbb{R}^n$ such that $Ax \leq b$ or there exists a vector $y \in \mathbb{R}^m$ such that $y \geq 0, y^T A = 0$ and $y^T b < 0$.

Proof.

We refer to the notation introduced in Definition 5.

$$C^{(n)} = \{y \in \mathbb{R}^m_+ \mid y^T A_{.j} = 0 \text{ for all } j = 1, \ldots, n\} = \{y \geq 0 \mid y^T A = 0\}.$$

$P^{(n)} = \{0 \leq y^T b \text{ for all } y \in C^{(n)}\}$. We conclude that

$$P \neq \emptyset \iff P^{(1)} \neq \emptyset \iff \ldots \iff P^{(n)} \neq \emptyset \iff y^T b \geq 0 \forall y \geq 0 \text{ with } y^T A = 0.$$

Either $P \neq \emptyset$ or $P = \emptyset$. Equivalently: either there exists

$$x \in \mathbb{R}^n \text{ such that } Ax \leq b \text{ or } y \in \mathbb{R}^m \text{ such that } y \geq 0,\ y^T A = 0 \text{ and } y^T b < 0.$$