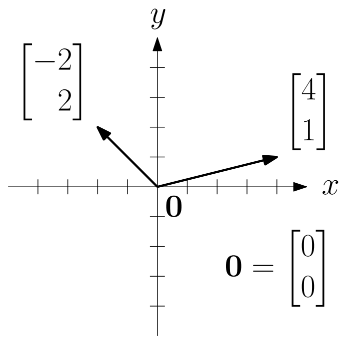
Week 0

Vectors and linear combinations (Section 1.1)

A vector is (for now) an element of $\mathbb{R}^m$, $m \in \mathbb{N} = \{0, 1, 2 \ldots\}$ (natural numbers).

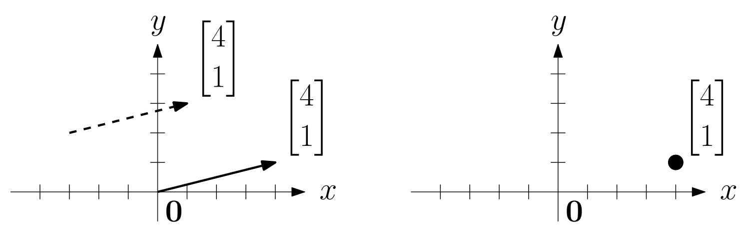
$\mathbb{R}^2$



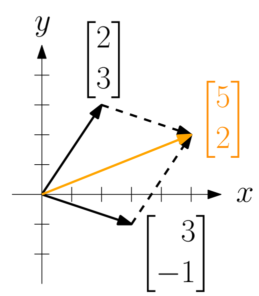
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

$\mathbb{R}^m$ (column vectors)

Drawing as arrow (movement) or point (location):



Vector addition: Combine the movements!



$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

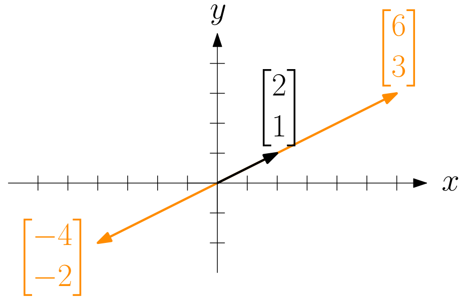
"parallelogram"

Definition 1.1: Let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \in \mathbb{R}^m. \text{ The vector } \mathbf{v}+\mathbf{w} := \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_m + w_m \end{bmatrix} \in \mathbb{R}^m \text{ is the sum of } \mathbf{v} \text{ and } \mathbf{w}.$$

More vectors: $\mathbf{u} + \mathbf{v} + \mathbf{w} := (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

Scalar multiplication: move λ times as far!



$$3\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix} \qquad (-2)\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-4\\-2\end{bmatrix}$$

Definition 1.3: Let

$$\mathbf{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_m\end{bmatrix} \in \mathbb{R}^m,\ \lambda \in \mathbb{R}. \text{ The vector } \lambda\mathbf{v} := \begin{bmatrix}\lambda v_1\\\lambda v_2\\\vdots\\\lambda v_m\end{bmatrix} \in \mathbb{R}^m \text{ is a } \textit{scalar multiple} \text{ of } \mathbf{v}.$$

Linear combination: all in one!

Definition 1.4: Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$, $\lambda, \mu \in \mathbb{R}$. Then

$$\lambda\mathbf{v} + \mu\mathbf{w} \in \mathbb{R}^m$$

is a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, then

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n$$

is a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

$$\mathbf{v} = \begin{bmatrix}2\\3\end{bmatrix}, \mathbf{w} = \begin{bmatrix}3\\-1\end{bmatrix}:$$

λ	μ	$\lambda\mathbf{v}$	$\mu\mathbf{w}$	$\lambda\mathbf{v} + \mu\mathbf{w}$
-3	2	$\begin{bmatrix}-6\\-9\end{bmatrix}$	$\begin{bmatrix}6\\-2\end{bmatrix}$	$\begin{bmatrix}0\\-11\end{bmatrix}$
1	-1	$\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}-3\\1\end{bmatrix}$	$\begin{bmatrix}-1\\4\end{bmatrix}$

Fact 1.5: Every vector $\mathbf{u} \in \mathbb{R}^2$ is a linear combination of

$$\mathbf{v} = \begin{bmatrix}2\\3\end{bmatrix}, \mathbf{w} = \begin{bmatrix}3\\-1\end{bmatrix}.$$

Proof. Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2.$$

Goal: find $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Two equations in two variables λ and μ:

$$\begin{aligned} 2\lambda + 3\mu &= u_1 \\ 3\lambda - 1\mu &= u_2. \end{aligned}$$

Add $3 \cdot$ (equation 2) to equation 1.

$$\begin{array}{rcrcrcr} 2\lambda & + & 3\mu & = & u_1 & & \\ 9\lambda & - & 3\mu & = & & & 3u_2 \\ \hline 11\lambda & & & = & u_1 & + & 3u_2 \end{array}$$

Solve for λ:

$$\lambda = \frac{u_1 + 3u_2}{11}.$$

Solve for μ (equation 1):

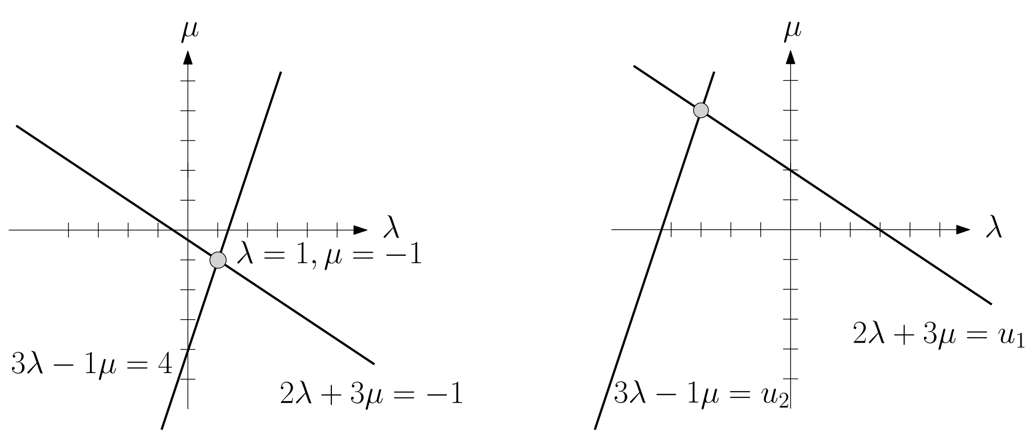
$$3\mu = u_1 - 2\lambda = u_1 - \frac{2u_1 + 6u_2}{11} = \frac{11u_1 - (2u_1 + 6u_2)}{11} = \frac{9u_1 - 6u_2}{11}.$$

Divide by 3:

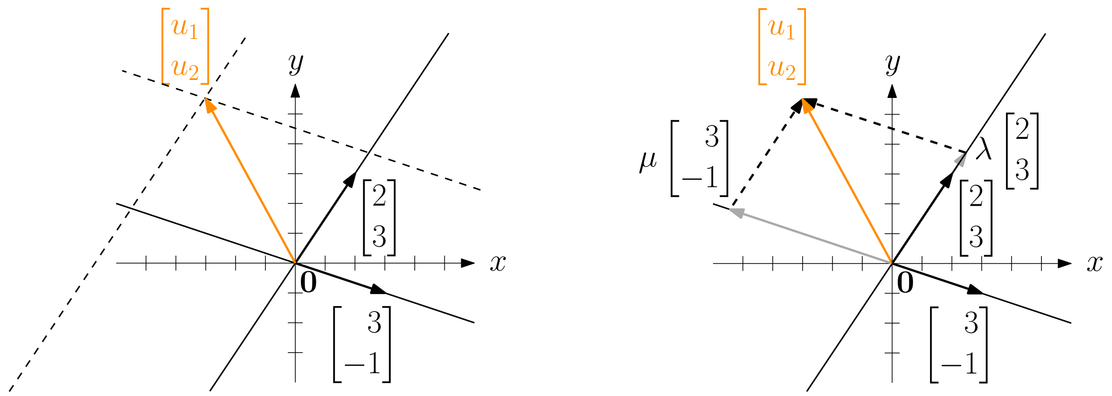
$$\mu = \frac{3u_1 - 2u_2}{11}.$$

□

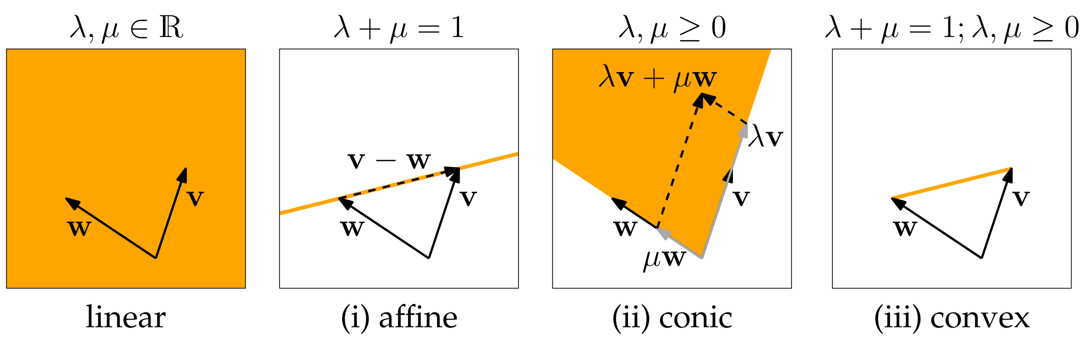
Row picture (Figure 1.8): $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and general $\mathbf{u}$



Column picture (Figure 1.9): construct the parallelogram!



Affine, conic, convex combinations: special linear combinations (Definition 1.7)



$$\begin{aligned} &\lambda\mathbf{v} + \mu\mathbf{w} \\ &= \lambda\mathbf{v} + (1-\lambda)\mathbf{w} \\ &= \mathbf{w} + \lambda(\mathbf{v} - \mathbf{w}) \end{aligned}$$