Lecture plan — Linear Algebra (401-0131-00L, HS24), ETH Zürich
Numbering of Sections, Definitions, Figures, etc. as in the Lecture Notes

Week 4

Gauss elimination (Section 3.2)

Algorithm for solving $A\mathbf{x} = \mathbf{b}$ with square matrix ($m \times m$).
Back substitution: if A is upper triangular

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 \\ 17 \\ 14 \end{bmatrix}$$

equation	before substitution	after substitution	solution
1	$2x_1 + 3x_2 + 4x_3 = 19$	$2x_1 + 11 = 19$	$x_1 = 4$
2	$5x_2 + 6x_3 = 17$	$5x_2 + 12 = 17$	$x_2 = 1$
3	$7x_3 = 14$		$x_3 = 2$

Case $m \times m$:
Equation $i = m, m-1, \ldots, 1$:

$$\sum_{j=i}^{m} a_{ij} x_j = b_i.$$

Already know $x_{i+1}, \ldots, x_m$:

$$x_i = \frac{b_i - \sum_{j=i+1}^{m} a_{ij} x_j}{a_{ii}}.$$

Needs $a_{ii} \neq 0$!

```
for (i = m-1; i >= 0; i--) {
    sum = b[i];
    for (j = i+1; j < m; j++)
        sum -= A[i][j] * x[j];
    x[i] = sum / A[i][i];
}
```

Back substitution code

Elimination: If A not upper triangular

- $A\mathbf{x} = \mathbf{b} \to U\mathbf{x} = \mathbf{c}$ (same solutions, U upper triangular)
- solve $U\mathbf{x} = \mathbf{c}$ with back substitution

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 19 \\ 55 \\ 50 \end{bmatrix}$$

Get rid of 4: subtract $2 \cdot$ (equation 1) from equation 2:

$$\begin{array}{rrcrcrcrcr} & \text{(equation 2)} & : & 4x_1 & + & 11x_2 & + & 14x_3 & = & 55 \\ - & 2\cdot\text{(equation 1)} & : & 4x_1 & + & 6x_2 & + & 8x_3 & = & 38 \\ \hline & \text{(equation 2')} & : & & & 5x_2 & + & 6x_3 & = & 17 \end{array}$$

$\rightarrow A'\mathbf{x} = \mathbf{b}'$:

$$A' = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} 19 \\ 17 \\ 50 \end{bmatrix}.$$

This was a *row subtraction*: linear transformation applied to all columns of A and $\mathbf{b}$:

$$T_{E_{21}}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 - 2x_1 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

E_{21}: *elimination matrix*

$A' = E_{21}A, \quad \mathbf{b}' = E_{21}\mathbf{b}$ — E_{21}: subtract $2\cdot$(row 1) from (row 2)

Can undo it:

$A = E'_{21}A', \quad \mathbf{b} = E'_{21}\mathbf{b}'$ — E'_{21}: add $2\cdot$(row 1) to (row 2)

$A\mathbf{x} = \mathbf{b}$ and $A'\mathbf{x} = \mathbf{b}'$ have the same solutions:

- If $A\mathbf{x} = \mathbf{b}$, then $A'\mathbf{x} = E_{21}A\mathbf{x} = E_{21}\mathbf{b} = \mathbf{b}'$
- If $A'\mathbf{x} = \mathbf{b}'$, then $A\mathbf{x} = E'_{21}A'\mathbf{x} = E'_{21}\mathbf{b}' = \mathbf{b}$

Column by column, eliminate all red entries ($\rightarrow$ upper triangular system from before):

fat number: the **pivot**

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 19 \\ 55 \\ 50 \end{bmatrix}$$

subtract $2\cdot$(row 1) from (row 2):

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix} \qquad E_{21}\mathbf{b} = \begin{bmatrix} 19 \\ 17 \\ 50 \end{bmatrix}$$

subtract $1\cdot$(row 1) from (row 3):

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{5} & 6 \\ 0 & 5 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \begin{bmatrix} 19 \\ 17 \\ 31 \end{bmatrix}$$

subtract $1\cdot$(row 2) from (row 3):

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \underbrace{E_{32}E_{31}E_{21}A}_{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & \mathbf{7} \end{bmatrix} \qquad \underbrace{E_{32}E_{31}E_{21}\mathbf{b}}_{\mathbf{c}} = \begin{bmatrix} 19 \\ 17 \\ 14 \end{bmatrix}$$

$\uparrow$ elimination matrices

done! Now back substitution…

Row exchanges: also undoable, solutions stay the same

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 8 & 17 \end{bmatrix} \qquad \mathbf{b} = \cdots$$

elimination in first column: $\downarrow \qquad \downarrow$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{0} & 6 \\ 0 & 5 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \cdots$$

pivot **0**: exchange (row 2) and (row 3): $\downarrow \qquad \downarrow$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \underbrace{P_{23}E_{31}E_{21}A}_{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{5} & 13 \\ 0 & 0 & 6 \end{bmatrix} \qquad \underbrace{P_{23}E_{31}E_{21}\mathbf{b}}_{\mathbf{c}} = \cdots$$

↑ permutation matrix

done!

Failure: no row exchange helps, give up for now!

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 3 & 17 \end{bmatrix} \qquad \mathbf{b} = \cdots$$

elimination in first column: $\downarrow \qquad \downarrow$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{0} & 6 \\ 0 & 0 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \cdots$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & \mathbf{0} \end{bmatrix}$$

we also fail here!

```
for (j = 0; j < m; j++) {
    // eliminate in column j
    if (A[j][j] == 0) {
        // zero pivot, try row exchange
        k = j+1;
        while (k < m && (A[k][j] == 0)) k++;
        if (k == m)
            return false; // no row exchange is possible, give up
        else {
            // exchange rows j and k ...
            row_A = A[j]; A[j] = A[k]; A[k] = row_A; // ... of A
            row_b = b[j]; b[j] = b[k]; b[k] = row_b; // ... of b
        }
    }
    // create zeros below A[j][j]
    for (i = j+1; i < m; i++) {
        // subtract c * row j from row i ...
        c = A[i][j] / A[j][j];
        A[i][j] = 0;
        for (k = j+1; k < m; k++)
            A[i][k] -= c * A[j][k]; // ... of A
        b[i] -= c * b[j];           // ... of b
    }
}
return true;
```

Elimination code

Success and failure
"Solutions stay the same", general case $m \times n$:

subtract $c \cdot$ (row j) from (row i) *row operation* exchange (row j) and (row k)

$$E_{ij} = \overbrace{\begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & \ddots & \\ & -c & & 1 & \\ & & & & \ddots \end{bmatrix}}^{m\times m} \begin{matrix} \\ \leftarrow j \\ \\ \leftarrow i \\ \\ \end{matrix} \qquad P_{jk} = \overbrace{\begin{bmatrix} \ddots & & & \\ & 0 & & 1 & \\ & & \ddots & \\ & 1 & & 0 & \\ & & & & \ddots \end{bmatrix}}^{m\times m} \begin{matrix} \\ \leftarrow j \\ \\ \leftarrow k \\ \\ \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ j & i \end{matrix} \qquad\qquad\qquad\qquad \begin{matrix} \uparrow & \uparrow \\ j & k \end{matrix}$$

elimination matrix *row operation matrix* permutation matrix

Lemma 3.3: Let $A\mathbf{x} = \mathbf{b}, A \in \mathbb{R}^{m\times n}$, $M \in \mathbb{R}^{m\times m}$ a row operation matrix, $A' = MA$ and $\mathbf{b}' = M\mathbf{b}$. $A\mathbf{x} = \mathbf{b}$ and $A'\mathbf{x} = \mathbf{b}'$ have the same solutions.

Proof. Undo M with M': add $c \cdot$ (row j) to (row i), or exchange (row j) and (row k) again:

$$A = M'A', \mathbf{b} = M'\mathbf{b}'.$$

$A\mathbf{x} = \mathbf{b} \Leftrightarrow A'\mathbf{x} = \mathbf{b}'$ follows as in 3×3 case. □

Corollary 3.4: Let $A \in \mathbb{R}^{m\times n}$, $M \in \mathbb{R}^{m\times m}$ a row operation matrix, $A' = MA$. A has linearly independent columns if and only if A' has linearly independent columns.

Proof. Lemma 3.3 with $\mathbf{b} = \mathbf{0}$: $A\mathbf{x} = \mathbf{0}$ and $A'\mathbf{x} = \mathbf{0}$ have the same solutions.

$$\begin{matrix} \text{Only } \mathbf{x} = \mathbf{0} \\ \text{Another } \mathbf{x} \end{matrix} : \text{both have linearly} \begin{matrix} \text{independent} \\ \text{dependent} \end{matrix} \text{columns (Observation 3.2).}$$ □

Theorem 3.5: Let $A\mathbf{x} = \mathbf{b}$ be a system of m linear equations in m variables. The following two statements are eqivalent.

(i) Gauss elimination succeeds.

(ii) The columns of A are linearly independent.

We prove (i)⇒(ii) and ¬(i)⇒ ¬(ii): if *not* (i), then *not* (ii). This is the *contraposition* of (ii)⇒(i) and logically equivalent.

Proof.
(i)⇒(ii): If Gauss elimination succeeds: $A \to$ upper triangular U with all $u_{jj} \neq 0$. U has linearly independent columns by Corollary 1.20 (iii): no column is a linear combination of the previous ones. We get (ii) by Corollary 3.4, repeatedly applied ($A \to A' \ldots \to U$).
¬(i)⇒ ¬(ii):If Gauss elimination fails in column j, $A \to A'$,

$$A' = \left[\begin{array}{c|c|c} U & \mathbf{v} & \cdots \\ \hline 0 & \mathbf{0} & \cdots \end{array}\right], \quad U \in \mathbb{R}^{(j-1)\times(j-1)} \text{ upper triangular, all } u_{\ell\ell} \neq 0, \mathbf{v} \in \mathbb{R}^{j-1}.$$

Construct $\mathbf{x} \neq 0 : A\mathbf{x} = \mathbf{0}$. Then A' has linearly dependent columns (Observation 3.2). This gives ¬(ii) by Corollary 3.4 ($A \to \ldots \to A'$). Construction:

$$x_{j+1}, x_{j+2}, \ldots, x_m = 0, \qquad U \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{j-1} \end{bmatrix}}_{\mathbf{y}} + x_j \mathbf{v} = \mathbf{0} \qquad \Rightarrow A\mathbf{x} = \mathbf{0}.$$

$$\uparrow$$

$x_j = -1,$ solve $U\mathbf{y} = \mathbf{v}$ (back substitution!)

□

Runtime: Count arithmetic steps $(-, \cdot, /)$!

Theorem 3.6 Let $A\mathbf{x} = \mathbf{b}$ be a system of m linear equations in m variables, $m \geq 1$. Gauss elimination with back substitution solves $A\mathbf{x} = \mathbf{b}$ (or gives up) with at most

$$g(m) = \frac{2}{3}m^3 + \frac{3}{2}m^2 - \frac{7}{6}m$$

arithmetic steps and therefore in time $O(m^3)$.

Proof. Count arithmetic steps in elimination: $e(m)$ steps

```
for (j = 0; j < m; j++) {
    // eliminate in column j
    if (A[j][j] == 0) {
        // zero pivot, try row exchange
        k = j+1;
        while (k < m && (A[k][j] == 0)) k++;
        if (k == m)
            return false; // no row exchange is possible, give up
        else {
            // exchange rows j and k ...
            row_A = A[j]; A[j] = A[k]; A[k] = row_A; // ... of A
            row_b = b[j]; b[j] = b[k]; b[k] = row_b; // ... of b
        }
    }
    // create zeros below A[j][j]
    for (i = j+1; i < m; i++) {
        // subtract c * row j from row i ...
        c = A[i][j] / A[j][j];                              1 /
        A[i][j] = 0;
        for (k = j+1; k < m; k++)
            A[i][k] -= c * A[j][k]; // ... of A             1 ·, 1 −
        b[i] -= c * b[j];           // ... of b             1 ·, 1 −
    }
}
return true;
```

Count arithmetic steps in back substitution: $b(m)$ steps

```
for (i = m-1; i >= 0; i--) {
    sum = b[i];
    for (j = i+1; j < m; j++)
        sum -= A[i][j] * x[j];                 1 ·, 1 −
    x[i] = sum / A[i][i];                      1 /
}
```

Counting formulas in the lecture notes! □

Dominating term: $\frac{2}{3}m^3$: Gauss elimination is very slow on large systems.
$O(m^3)$: 3 nested loops in elimination (lines 1, 16, 20)

Inverse matrices (Section 3.3)

Row operation matrix M:

do	undo	do & undo	$A = I$	undo & do	$A' = I$
$A' = MA$	$A = M'A'$	$A = M'\underbrace{MA}_{A'}$	$M'M = I$	$A' = M\underbrace{M'A'}_{A}$	$MM' = I$

Definition 3.7 Let M be an $m \times m$ matrix. M is called *invertible* if there exists an $m \times m$ matrix M^{-1} (called the *inverse* of M) such that

$$MM^{-1} = M^{-1}M = I.$$

Case 1×1:

$$M = \begin{bmatrix} a \end{bmatrix} \quad \Rightarrow \quad M^{-1} = \begin{bmatrix} \frac{1}{a} \end{bmatrix} \quad \text{(if } a \neq 0\text{).}$$

Case 2×2**:** check it!

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \Rightarrow \quad M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(if } ad - bc \neq 0\text{).}$$

Inverse is unique:
Lemma 3.8: Let M be an $m \times m$ matrix with two inverses A and B. Then $A = B$.

Proof. $A = IA = (BM)A = B(MA) = BI = B$. □

Lemma 3.9: Let A and B be invertible $m \times m$ matrices. Then AB is invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof sketch. Undo in reverse order (put socks on, then shoes; take shoes off, then socks).
Lemma 3.10: Let A be an invertible $m \times m$ matrix. Then the transpose $A^\top$ is invertible, and

$$\left(A^\top\right)^{-1} = \left(A^{-1}\right)^\top.$$

The Inverse Theorem: "good" matrices
Theorem 3.11: Let A be an $m \times m$ matrix. The following statements are equivalent.

(i) A is invertible.

(ii) For every $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$.

(iii) The columns of A are linearly independent.

Success of Gauss elimination $\Leftrightarrow$ (iii), Theorem 3.5.
Proof plan:

$$\begin{array}{ccc} \text{(i)} & \Rightarrow & \text{(ii)} \\ \Uparrow & & \Downarrow \\ \text{(ii)} & \Leftarrow & \text{(iii)} \end{array}$$

Proof.
(i) $\Rightarrow$ (ii): if A is invertible, $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$:

$$A(\underbrace{A^{-1}\mathbf{b}}_{\mathbf{x}}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}.$$

Uniqueness: take any $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$:

$$\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

(ii) $\Rightarrow$ (iii): if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, then also for $\mathbf{b} = \mathbf{0}$. By Observation 3.2, the columns of A are linearly independent.
(iii) $\Rightarrow$ (ii): Let $\mathbf{x}, \mathbf{x}'$ be two solutions: $A\mathbf{x} = A\mathbf{x}' = \mathbf{b}$. Then $A(\mathbf{x} - \mathbf{x}') = \mathbf{0}$. If the columns of A are linearly independent, then $\mathbf{x} - \mathbf{x}'$ must be $\mathbf{0}$ (Observation 3.2), so $\mathbf{x} = \mathbf{x}'$.
(ii) $\Rightarrow$ (i): If $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$, we find $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^m$ such that

$$A\mathbf{v}_1 = \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{e}_1}, A\mathbf{v}_2 = \underbrace{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{e}_2}, \ldots, A\mathbf{v}_m = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{e}_m} \quad \Rightarrow \quad A \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \\ | & | & & | \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{I}.$$

Still need $BA = I$:

- $AI = IA = (AB)A = A(BA)$, so $A(I - BA) = 0$.
- $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$: columns of $I - BA$. Then $A(I - BA) = 0$ means $A\mathbf{w}_j = \mathbf{0}$ for all j.
- The columns of A are linearly independent by (ii) $\Rightarrow$ (iii). Hence $\mathbf{w}_j = \mathbf{0}$ for all j: by Observation 3.2, $\mathbf{0}$ is the only solution of $A\mathbf{x} = \mathbf{0}$.
- All columns of $I - BA$ are $\mathbf{0}$, so $BA = I$.

$\square$

Exercise 3.12: For *all* A, B: If $AB = I$, then $BA = I$, so we only need one condition in Definition 3.7 of the inverse.