Week 7

Computing the fundamental subspaces (Section 4.3)

Computing a vector space: compute a basis!
Now do this for the *fundamental subspaces* of a matrix A:

- $\mathbf{C}(A)$ (column space),
- $\mathbf{R}(A)$ (row space),
- $\mathbf{N}(A)$ (nullspace)
- $\mathbf{LN}(A) = \mathbf{N}(A^\top)$ (left nullspace; Section 4.3.4)

Tool: Gauss-Jordan elimination

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \quad \to \quad R_0 = \underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{REF(1,3)}}. \quad \text{(RE, running example)}$$

Column space:

$$\mathbf{C}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m \quad \text{(subspace by Lemma 4.11)}$$

Theorem 4.25: Let A be an $m \times n$ matrix, and let R_0 in $\mathrm{REF}(j_1, j_2, \ldots, j_r)$ be the result of Gauss-Jordan elimination on A. Then A has independent columns $j_1, j_2, \ldots, j_r$, and these form a basis of the column space $\mathbf{C}(A)$. Hence

$$\dim(\mathbf{C}(A)) = r = \mathbf{rank}(A).$$

Proof. Independent columns: basis of $\mathbf{C}(A)$ (Lemma 4.17); using Gauss-Jordan elimination, we can compute their positions ($j_1, j_2, \ldots, j_r$, by Lemma 3.22). □

RE:

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \quad \text{(columns } 1, 3 \text{ of } A\text{)}.$$

Row space:

$$\mathbf{R}(A) = \mathbf{C}(A^\top) \subseteq \mathbb{R}^n.$$

Lemma 4.11 applied to $A^\top$:

Corollary 4.26: Let A be an $m \times n$ matrix. Then $\mathbf{R}(A)$ is a subspace of $\mathbb{R}^n$.

Main insight: A and $R_0 = MA$ (Theorem 3.21) have the same row space!

Lemma 4.27: Let A be an $m \times n$ matrix, M an invertible $m \times m$ matrix.
Then $\mathbf{R}(A) = \mathbf{R}(MA)$.

Proof.

$$\begin{array}{ccc} \mathbf{R}(A) & & \mathbf{R}(MA) \\ \| & & \| \\ \mathbf{C}(\underbrace{B}_{A^\top}) & \stackrel{!}{=} & \mathbf{C}(\underbrace{\underbrace{B}_{A^\top}\underbrace{N}_{M^\top}}_{(MA)^\top}) \end{array}$$

$$\begin{array}{c} \mathbf{v} \in \mathbf{C}(B) \\ \Updownarrow \\ \mathbf{v} = B\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^m \\ \Uparrow \qquad\qquad \Downarrow \\ \mathbf{v} = BN\mathbf{y} \text{ for some } \mathbf{y} \in \mathbb{R}^m \\ \Updownarrow \\ \mathbf{v} \in \mathbf{C}(BN). \end{array} \qquad \leftarrow \mathbf{y} := N^{-1}\mathbf{x} \Leftrightarrow \mathbf{x} := N\mathbf{y}$$

□

Theorem 4.28: Let A be an $m \times n$ matrix, and let R_0 in $\mathrm{REF}(j_1, j_2, \ldots, j_r)$ be the result of Gauss-Jordan elimination on A. Then the first r rows of R_0 form a basis of the row space $\mathbf{R}(A)$, and

$$\dim(\mathbf{R}(A)) = r = \mathbf{rank}(A).$$

Proof. $R_0 = MA$, $\mathbf{R}(A) = \mathbf{R}(R_0)$. Basis of $\mathbf{R}(R_0)$ can be read off: R_0 ends with $m - r$ zero rows ⇒ the first r rows span $\mathbf{R}(R_0)$ and are linearly independent (private nonzeros at the "downward steps"). By Lemma 3.22, $r = \mathbf{rank}(A)$. □

RE:

$$B = \left\{ \begin{bmatrix} 1 & 2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -2 \end{bmatrix} \right\} \quad \text{(rows } 1, 2 \text{ of } R_0\text{)}$$

Because $\underbrace{\dim(\mathbf{C}(A))}_{\mathbf{rank}(A)} \underbrace{= \dim(\mathbf{R}(A))}_{\text{Theorem 4.28}} \underbrace{= \overbrace{\dim(\mathbf{C}(A^\top))}^{\mathbf{rank}(A^\top)}}_{\text{Def. of row space}}$, row rank equals column rank:

Theorem 4.29: Let A be an $m \times n$ matrix. Then $\mathbf{rank}(A) = \mathbf{rank}(A^\top)$.

Nullspace (Definition 4.31):

$$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n.$$

Lemma 4.32: Proof similar to column space

Let A be an $m \times n$ matrix. Then $\mathbf{N}(A)$ is a subspace of $\mathbb{R}^n$.

$R_0 = MA$ helps again:

Lemma 4.33: Let A be an $m \times n$ matrix, M an invertible $m \times m$ matrix.
Then $\mathbf{N}(A) = \mathbf{N}(MA)$.

Proof.

$$\begin{array}{ccccccc} \mathbf{x} \in \mathbf{N}(A) & \Leftrightarrow & A\mathbf{x} = \mathbf{0} & \Rightarrow & MA\mathbf{x} = M\mathbf{0} & & \\ & & \Uparrow & & \Downarrow & & \\ & & M^{-1}MA\mathbf{x} = M^{-1}\mathbf{0} & \Leftarrow & MA\mathbf{x} = \mathbf{0} & \Leftrightarrow & \mathbf{x} \in \mathbf{N}(MA). \end{array}$$

□

Basis of $\mathbf{N}(A) = \mathbf{N}(R_0)$ (in RE):

$\mathbf{N}(R_0) = \mathbf{N}(R)$ (zero rows don't affect the nullspace), $R = \underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}}_{\text{RREF(1,3)}}$.

$R\mathbf{x} = \mathbf{0}$ becomes: $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}}_{\mathbf{x}(I)} + \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}}_{\mathbf{x}(Q)} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{x}(I) = -Q\mathbf{x}(Q).$

Choose *free variables* $\mathbf{x}(Q)$, read off *basic variables* $\mathbf{x}(I)$.

Basis obtained from *special solutions*: choose the standard unit vectors for $\mathbf{x}(Q)$:

		special solutions	
free variables	$\underbrace{\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}}_{\mathbf{x}(Q)}$	$\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1(Q)}$	$\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2(Q)}$
basic variables	$\underbrace{\begin{bmatrix} -2 & -3 \\ 0 & 2 \end{bmatrix}}_{-Q} \underbrace{\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}}_{\mathbf{x}(Q)} = \underbrace{\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}}_{\mathbf{x}(I)}$	$\underbrace{\begin{bmatrix} -2 \\ 0 \end{bmatrix}}_{\mathbf{v}_1(I)}$	$\underbrace{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}_{\mathbf{v}_2(I)}$
nullspace equation	$\mathbf{0} = \underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}}$	$\underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1}$	$\underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$

General picture, $R \in \mathrm{RREF}(j_1, j_2, \ldots, j_r)$:

$n - r$ free variables $\mathbf{x}(Q)$, r basic variables $\mathbf{x}(I)$, basis has $n - r$ elements (Lemma 4.34).

Theorem 4.35: Let A be an $m \times n$ matrix, and let R_0 in $\mathrm{REF}(j_1, j_2, \ldots, j_r)$ be the result of Gauss-Jordan elimination on A. Let R in $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$ be the submatrix of R_0 consisting of the first r rows. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots . \mathbf{v}_{n-r}$ as constructed in Lemma 4.34 form a basis of $\mathbf{N}(A) = \mathbf{N}(R_0) = \mathbf{N}(R)$, and therefore,

$$\dim(\mathbf{N}(A)) = n - r = n - \mathbf{rank}(A).$$

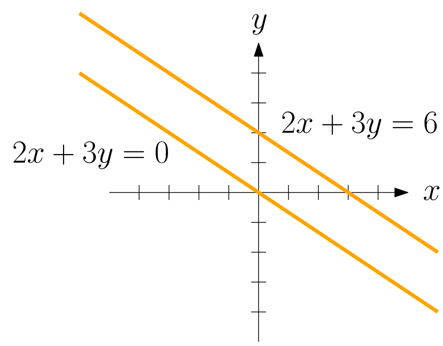
Summary (for bases: Figure 4.5):

subspace		
name	symbol	dimension
Column space	$\mathbf{C}(A)$	r
Row space	$\mathbf{R}(A)$	r
Nullspace	$\mathbf{N}(A)$	$n - r$
Left nullspace	$\mathbf{LN}(A) = \mathbf{N}(A^\top)$	$m - r$

The solution space of $A\mathbf{x} = \mathbf{b}$ (Definition 4.39):

$$\mathbf{Sol}(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} \subseteq \mathbb{R}^n$$

If $\mathbf{b} \neq \mathbf{0}$, $\mathbf{Sol}(A, \mathbf{b})$ is not a subspace (missing $\mathbf{0}$).



$\mathbf{Sol}(A, b)$ for $A = [2 \quad 3]$, $\mathbf{b} = [6]$ and $\mathbf{b} = [0]$

Theorem 4.40: Let A be an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{x}$ be some solution of $A\mathbf{x} = \mathbf{b}$. Then

$$\mathbf{Sol}(A, \mathbf{b}) = \{\mathbf{x} + \mathbf{v} : \mathbf{v} \in \mathbf{N}(A)\}.$$

"Basis" of $\mathbf{Sol}(A, \mathbf{b})$: some solution $\mathbf{x}$ (if it exists) and a basis of $\mathbf{N}(A)$.

Proof.

(i) $\mathbf{Sol}(A, \mathbf{b}) \ni \mathbf{y} = \mathbf{x} + \underbrace{\mathbf{v}}_{\in \mathbf{N}(A)}$: $\quad \mathbf{y} = \mathbf{x} + \underbrace{(\mathbf{y} - \mathbf{x})}_{\mathbf{v}}, \quad A\mathbf{v} = A\mathbf{y} - A\mathbf{x} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$

(ii) $\mathbf{y} = \mathbf{x} + \underbrace{\mathbf{v}}_{\in \mathbf{N}(A)} \in \mathbf{Sol}(A, \mathbf{b})$: $\quad A\mathbf{y} = A\mathbf{x} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$ □

If there is a solution, $\mathbf{Sol}(A, \mathbf{b})$ has "dimension" $n - r = \dim(\mathbf{N}(A))$.

$r = m$: there is always a solution (Lemma 4.41). $\quad r < m$: "typically", no solution.